

Problem Set 10: Bosons, fermions, and anyons
Graduate Quantum I
Physics 6572

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Potentially useful reading

Sakurai and Napolitano, sections 7.1-7.2 (identical particles)

Schumacher & Westmoreland 14.3 (identical particles)

Weinberg section 4.5 (bosons & fermions)

Group Theory, by D. D. Vvedensky,

<http://www.cmth.ph.ic.ac.uk/people/d.vvedensky/groups/Chapter8.pdf>

Sethna [ignore temperature] sections 7.3 (bosons and fermions), 7.4 (non-interacting), 7.5 (Maxwell-Boltzmann), 7.6 (free particles, black body, Bose condensation), 7.7 (Fermi gas).

10.1 **Bosons are gregarious: superfluids and lasers.** (Quantum, Optics, Atomic physics) ③

Adding a particle to a Bose condensate. Suppose we have a non-interacting system of bosonic atoms in a box with single-particle orthonormal eigenstates ψ_n . Suppose the system begins with all N bosons in a state ψ_0 (a “Bose condensed state”), so

$$\Psi_N^{[0]}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \psi_0(\mathbf{r}_1) \cdots \psi_0(\mathbf{r}_N). \quad (1)$$

Suppose a new particle is gently injected into the system, into an equal superposition of the M lowest single-particle states.¹ That is, if it were injected into an empty box, it would start in state

$$\begin{aligned} \phi(\mathbf{r}_{N+1}) = \frac{1}{\sqrt{M}} & (\psi_0(\mathbf{r}_{N+1}) + \psi_1(\mathbf{r}_{N+1}) \\ & + \dots + \psi_{M-1}(\mathbf{r}_{N+1})). \end{aligned} \quad (2)$$

The state $\Phi(\mathbf{r}_1, \dots, \mathbf{r}_{N+1})$ after the particle is inserted into the non-interacting Bose condensate is given by symmetrizing the product function $\Psi_N^{[0]}(\mathbf{r}_1, \dots, \mathbf{r}_N)\phi(\mathbf{r}_{N+1})$

$$\Psi_{\text{sym}}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{N+1}) = (\text{normalization}) \sum_P \Psi_N^{[0]}(\mathbf{r}_{P_1}, \mathbf{r}_{P_2}, \dots, \mathbf{r}_{P_N})\phi(\mathbf{r}_{P_{N+1}}). \quad (3)$$

¹For free particles in a cubical box of volume V , injecting a particle at the origin $\phi(\mathbf{r}) = \delta(\mathbf{r})$ would be a superposition of *all* plane-wave states of equal weight, $\delta(\mathbf{r}) = (1/V) \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$. (In second-quantized notation, $a^\dagger(\mathbf{x} = 0) = (1/V) \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger$.) We ‘gently’ add a particle at the origin by restricting this sum to low-energy states. This is how quantum tunneling into condensed states (say, in Josephson junctions or scanning tunneling microscopes) is usually modeled.

So, for example, if $M = 2$ and $N = 1$,

$$\begin{aligned}\Psi_{\text{sym}}(\mathbf{r}_1, \mathbf{r}_2) &= \mathcal{S}[\psi_0(\mathbf{r}_1)(\psi_0(\mathbf{r}_2) + \psi_1(\mathbf{r}_2))/\sqrt{2}] \\ &= (\text{normalization})[\psi_0(\mathbf{r}_1)(\psi_0(\mathbf{r}_2) + \psi_1(\mathbf{r}_2)) + \psi_0(\mathbf{r}_2)(\psi_0(\mathbf{r}_1) + \psi_1(\mathbf{r}_1))] \quad (4) \\ &= (\text{normalization})[2\psi_0(\mathbf{r}_1)\psi_0(\mathbf{r}_2) + \psi_0(\mathbf{r}_1)\psi_1(\mathbf{r}_2) + \psi_0(\mathbf{r}_2)\psi_1(\mathbf{r}_1)].\end{aligned}$$

Since the ψ_n are orthonormal, the integral of the term in brackets squared is $2^2 + 1^2 + 1^2 = 6$, so the normalization is $1/\sqrt{6}$. The probability of the new particle entering state $n = 0$, $4/6$, is twice the net probability $2/6$ of the new particle entering state $n = 1$.

(a) Calculate the symmetrized initial state of the system with the injected particle for general M and N . Show that the ratio of the probability that the new boson enters the ground state ψ_0 is enhanced over that of its entering a particular empty state² (ψ_m for $0 < m < M$) by a factor $N + 1$.

So, if a macroscopic number of bosons are in one single-particle eigenstate, a new particle will be much more likely to add itself to this state than to any of the microscopically populated states.

Notice that nothing in your analysis depended on ψ_0 being the lowest energy state. If we started with a macroscopic number of particles in a single-particle state with wavevector \mathbf{k} (that is, a superfluid with a supercurrent in direction \mathbf{k}), new added particles, or particles scattered by inhomogeneities, will preferentially enter into that state. This is an alternative approach to understanding the persistence of supercurrents, complementary to the topological approach (Exercise 2).

Adding a photon to a laser beam. This ‘chummy’ behavior between bosons is also the principle behind lasers.³ A laser has N photons in a particular mode. An atom in an excited state emits a photon. The photon it emits will prefer to join the laser beam than to go off into one of its other available modes by a factor $N + 1$. Here the N represents *stimulated emission*, where the existing electromagnetic field pulls out the energy from the excited atom, and the $+1$ represents *spontaneous emission* which occurs even in the absence of existing photons.

Imagine a single atom in a state with excitation energy E and decay rate Γ , in a cubical box of volume V with periodic boundary conditions for the photons. By the energy-time uncertainty principle, $\langle \Delta E \Delta t \rangle \geq \hbar/2$, the energy of the atom will be uncertain by an amount $\Delta E \propto \hbar\Gamma$. Assume for simplicity that, in a cubical box without pre-existing photons, the atom would decay at an equal rate into any mode in the range $E - \hbar\Gamma/2 < \hbar\omega < E + \hbar\Gamma/2$.

(b) Assuming a large box and a small decay rate Γ , find a formula for the number of modes M per unit volume V per unit energy E in the box (the density of states). How

²More precisely, calculate the ratio of the probability of being in the many-body ground state (all particles in state ψ_0) to the probability of injecting into the many-body state with one boson in the state ψ_m and the rest in ψ_0 .

³Laser is an acronym for ‘light amplification by the stimulated emission of radiation’.

many states are competing for the photon emitted from our atom, for a laser with wavelength $\lambda = 619 \text{ nm}$ and line-width $\Gamma = 10^4 \text{ rad/s}$. (Hint: The eigenstates are plane waves, with two polarizations per wavevector. Using periodic boundary conditions, one can derive the density of states. This is a standard calculation, so you can look up the answer to check it.)

Assume the laser is already in operation, so there are N photons in the volume V of the lasing material, all in one plane-wave state (a *single-mode* laser).

(c) Using your result from part (a), give a formula for the number of photons per unit volume N/V there must be in the lasing mode for the atom to have 50% likelihood of emitting into that mode.

The main task in setting up a laser is providing a population of excited atoms. Amplification can occur if there is a *population inversion*, where the number of excited atoms is larger than the number of atoms in the lower energy state (definitely a non-equilibrium condition). This is made possible by *pumping* atoms into the excited state by using one or two other single-particle eigenstates.

10.2 Superfluid order and vortices. (Quantum, Condensed matter) ③

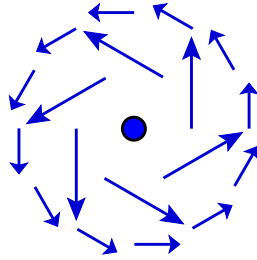


Fig. 1 Superfluid vortex line. Velocity flow $\mathbf{v}(\mathbf{x})$ around a superfluid vortex line.

Superfluidity in helium is closely related to Bose condensation of an ideal gas; the strong interactions between the helium atoms quantitatively change things, but many properties are shared. In particular, we describe the superfluid in terms of a complex number $\psi(\mathbf{r})$, which we think of as a wavefunction which is occupied by a large fraction of all the atoms in the fluid.

(a) If N non-interacting bosons all reside in the same single-particle state with wavefunction $\zeta(\mathbf{r})$, write an expression for the net current density $J(\mathbf{r})$.⁴ Write the complex field $\zeta(\mathbf{r})$ in terms of an amplitude and a phase, $\zeta(\mathbf{r}) = |\zeta(\mathbf{r})| \exp(i\phi(\mathbf{r}))$. We write the

⁴You can use the standard quantum mechanics single-particle expression $J = (\hbar/2m)(\zeta\nabla\zeta^* - \zeta^*\nabla\zeta)$ and multiply by the number of particles, or you can use the many-particle formula $J(\mathbf{r}) = (\hbar/2m) \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N \sum_\ell \delta(\mathbf{r}_\ell - \mathbf{r})(\Psi\nabla_\ell\Psi^* - \Psi^*\nabla_\ell\Psi)$ and substitute in the condensate wavefunction $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_n \zeta(\mathbf{r}_n)$.

superfluid density as $n_s = N|\zeta|^2$. Give the current J in terms of ϕ and n_s . What is the resulting superfluid velocity, $v = J/n_s$? (It should be independent of n_s .)

The Landau order parameter in superfluids $\psi(\mathbf{r})$ is traditionally normalized so that the amplitude is the square root of the superfluid density; in part (a), $\psi(\mathbf{r}) = \sqrt{N}\zeta(\mathbf{r})$.

In equilibrium statistical mechanics, the macroscopically occupied state is always the ground state, which is real and hence has no current. We can form non-equilibrium states, however, which macroscopically occupy other quantum states. For example, an experimentalist might cool a container filled with helium while it is moving; the ground state in the moving reference frame has a current in the unmoving laboratory frame. More commonly, the helium is prepared in a rotating state.

(b) Consider a torus filled with an ideal Bose gas at $T = 0$ with the hole along the vertical axis; the superfluid is condensed into a state which is rotating around the hole. Using your formula from part (a) and the fact that $\phi + 2n\pi$ is indistinguishable from ϕ for any integer n , show that the circulation $\oint v \cdot d\mathbf{r}$ around the hole is quantized. What is the quantum of circulation?

Superfluid helium cannot swirl except in quantized units! Notice that you have now explained why superfluids have no viscosity. The velocity around the torus is quantized, and hence it cannot decay continuously to zero; if it starts swirling with non-zero n around the torus, it must swirl forever.⁵ This is why we call them superfluids.

In bulk helium this winding number labels line defects called *vortex lines*.

10.3 Fourier series and group representations. (Math) ③

In class, we focused on finite-dimensional group representations for finite groups. In quantum mechanics, the most useful symmetries are often continuous, and Hilbert space is infinite dimensional. With some small modifications, all of our results can go through to the continuous case.

Here we apply group representation theory to the continuous rotations in the plane, $SO(2)$. Let $g_\phi \in SO(2)$ be the rotation by angle ϕ .⁶

(a) Show that every different g_ϕ is in its own conjugacy class. (This is true for any commutative group.)

Thus we may label the conjugacy classes by the angle ϕ .

Consider the action of g_ϕ on a function $f(\theta)$:

$$R(g_\phi) : f(\theta) \rightarrow f(\theta - \phi). \quad (5)$$

⁵Or at least until a dramatic event occurs which changes n , like a vortex line passing across the torus, demanding an activation energy proportional to the width of the torus. See also Exercise (7.9) in my book.

⁶This exercise is mostly about understanding the definitions. If you find resources on the Web or elsewhere that are helpful, just properly acknowledge them. In particular, I found <http://www.cmth.ph.ic.ac.uk/people/d.vvedensky/groups/Chapter8.pdf> which discusses the application of group reps to $SO(2)$. No guarantees that my conventions agree with those in the literature, though.

Here θ represents a point on a circle, the complex function $f(\theta)$ is a vector in the Hilbert space of complex functions⁷ on the circle, and $R(g_\phi)$ is a linear mapping of that function space into itself.⁸

(b) Show that, for any non-negative integer m , that the two-dimensional space spanned by the basis $\{\cos(m\theta), \sin(m\theta)\}$ is an invariant subspace under $SO(2)$. Give the explicit 2×2 matrix for $R(g_\phi)$ acting on this subspace in this basis. What is the character $\chi(\phi)$ of this representation? (Hint: Use the angle addition formulas. Check that the character of the identity is the dimension of the representation.)

In the space of complex functions on the circle, this two-dimensional representation is not irreducible. It can be decomposed into two invariant subspaces, with bases $\{e^{im\theta}\}$ and $\{e^{-im\theta}\}$.

(c) What is the character of the ‘ m ’-representation given by the one-dimensional invariant subspace of multiples of $\{e^{im\theta}\}$?

Thus the ‘character table’ for $SO(2)$ would have an infinite number of rows (one for each integer $\pm m$) and a continuous infinity of columns (one for each angle ϕ).

For finite groups, we decomposed representations into irreducible pieces using the ‘little’ orthogonality theorem: for any two irreducible representations α and β , the sum over group elements $\sum_{g \in G} \chi^{(\alpha)}(g)\chi^{(\beta)}(g)^* = o(G)\delta_{\alpha\beta}$, where $o(G)$ is the number of elements of the group. For continuous groups, the sum must be replaced by an integral over the group,⁹ and the number of elements of the group replaced by the ‘volume’ of the group. For two-dimensional rotations, we find

$$\int_0^{2\pi} d\phi \chi^{(\alpha)}(\phi)\chi^{(\beta)}(\phi)^* = 2\pi\delta_{\alpha\beta}. \quad (6)$$

(d) Show that the characters of your irreducible representations from part (c) satisfy the orthogonality relation 6. Is the character of your reducible representation in part (b) orthogonal to all the irreducible representations? Use the little orthogonality relation explicitly to decompose this reducible representation into its irreducible components.

For finite-dimensional representations of finite groups, we knew that any representation could be decomposed into irreducible representations: that is, any general vector could be written as a sum of vectors in the different invariant subspaces. For example, in Alemi’s analysis of vibrations in a triangular molecule, he found the normal modes by using a projection operator

$$P^{(\alpha)} = (f^{(\alpha)}/o(G)) \sum_{g \in G} \chi^{(\alpha)}(g)^* R(g). \quad (7)$$

⁷Particularly, L^2 functions on the circle.

⁸In the past, we viewed group representations as mappings of the group into spaces of matrices that preserve multiplication. But matrices are just linear transformations of vectors; here we are using infinite dimensional vectors instead. Thus $R(g)$ is a linear map taking a function to another function.

⁹For $SO(2)$, this is just an integral over ϕ . More generally, and in particular for $SO(3)$, you have an extra factor in the integral (the *Haar measure*).

Here¹⁰ $f^{(\alpha)}$ is the dimension of the representation α .

For example, any random deformation of the molecule, when averaged over the group, gave a uniform dialation of the triangle. This dialation is invariant under triangular symmetries – so it transforms under the representation A_1 . Since $\chi^{(A_1)}(g) \equiv 1$, this is just what eqn (7) suggests. When Alex multiplied by the characters of the two-dimensional representation E (using $P^{(E)}$ in eqn 7), though, he discovered a different normal mode that was doubly degenerate.

Let us return now to our infinite-dimensional space of complex functions on the circle, to connect our irreducible representation decomposition with the theory of Fourier series. For our continuous group $SO(2)$, the corresponding projection operator is

$$P^{(\alpha)} = (1/2\pi) \int_0^{2\pi} d\phi \chi^{(\alpha)}(\phi)^* R(g_\phi). \quad (8)$$

Let $f(\theta)$ be a particular complex function on the circle. Let $R(g_\phi)$ be defined on the function as in eqn (5).

(e) Show that the projection operator in eqn (8), using the ‘ m ’ representation of part (c), takes $f(\theta)$ into a coefficient times the basis vector for that representation. How is the coefficient related to the Fourier series coefficient¹¹ \tilde{f}_m for f ?

If we sum the projections of a vector into all the invariant subspaces, we should get the vector back again.

(f) Write the sum of the projections of $f(\theta)$ over all the irreducible representations of $SO(2)$. Do you recognize this formula? Is it equal to the function $f(\theta)$?

This is the underlying mathematical reason why one can expand periodic functions in Fourier series.

10.4 Anyons. (Statistics) ③

Frank Wilczek, “Quantum mechanics of fractional-spin particles”, *Phys. Rev. Lett.* **49**, 957 (1982).

Steven Kivelson, Dung-Hai Lee, and Shou-Cheng Zhang, “Electrons in Flatland”, *Scientific American*, March 1996.

In quantum mechanics, identical particles are truly indistinguishable (Fig. 2). This means that the wavefunction for these particles must return to itself, up to an overall phase, when the particles are permuted:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots) = \exp(i\chi) \Psi(\mathbf{r}_2, \mathbf{r}_1, \dots). \quad (9)$$

¹⁰Alex didn’t bother with the factor ($f^{(\alpha)}/o(G)$), since he just wanted a vector in the subspace. We want to make the sum over representations α equal to the original function. Alex also, I think, missed the complex conjugate (but all his characters were real).

¹¹There are many different conventions for Fourier series. Clearly state which one you are using.

where \dots represents potentially many other identical particles.

We can illustrate this with a peek at an advanced topic mixing quantum field theory and relativity. Here is a scattering event of a photon off an electron, viewed in two reference frames; time is vertical, a spatial coordinate is horizontal. On the left we see two ‘different’ electrons, one which is created along with an anti-electron or positron e^+ , and the other which later annihilates the positron. On the right we see the same event viewed in a different reference frame; here there is only one electron, which scatters two photons. (The electron is *virtual*, moving faster than light, between the collisions; this is allowed in intermediate states for quantum transitions.) The two electrons on the left are not only indistinguishable, they are the *same particle*! The antiparticle is also the electron, traveling backward in time.¹²

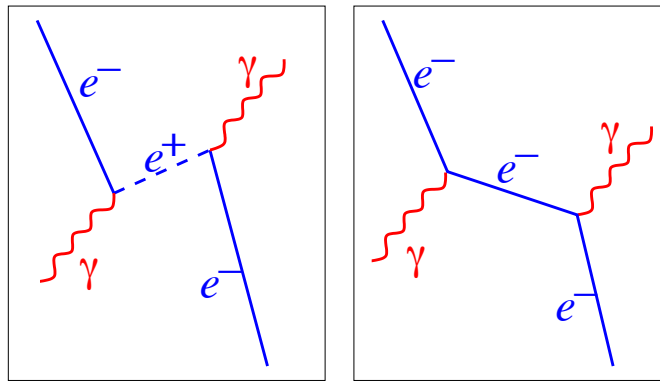


Fig. 2 Feynman diagram: identical particles.

In three dimensions, χ must be either zero or π , corresponding to bosons and fermions. In two dimensions, however, χ can be anything: *anyons* are possible! Let’s see how this is possible.

In a two-dimensional system, consider changing from coordinates $\mathbf{r}_1, \mathbf{r}_2$ to the center-of-mass vector $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, the distance between the particles $r = |\mathbf{r}_2 - \mathbf{r}_1|$, and the angle ϕ of the vector between the particles with respect to the \hat{x} axis. Now consider permuting the two particles counter-clockwise around one another, by increasing ϕ at fixed r . When $\phi = 180^\circ \equiv \pi$, the particles have exchanged positions, leading to a

¹²This idea is due to Feynman’s thesis advisor, John Archibald Wheeler. As Feynman quotes in his Nobel lecture, *I received a telephone call one day at the graduate college at Princeton from Professor Wheeler, in which he said, “Feynman, I know why all electrons have the same charge and the same mass.” “Why?” “Because, they are all the same electron!” And, then he explained on the telephone, “suppose that the world lines which we were ordinarily considering before in time and space - instead of only going up in time were a tremendous knot, and then, when we cut through the knot, by the plane corresponding to a fixed time, we would see many, many world lines and that would represent many electrons, except for one thing. If in one section this is an ordinary electron world line, in the section in which it reversed itself and is coming back from the future we have the wrong sign to the proper time - to the proper four velocities - and that’s equivalent to changing the sign of the charge, and, therefore, that part of a path would act like a positron.”*

boundary condition on the wavefunction

$$\Psi(\mathbf{R}, r, \phi, \dots) = \exp(i\chi)\Psi(\mathbf{R}, r, \phi + \pi, \dots). \quad (10)$$

Permuting them counter-clockwise (backward along the same path) must then¹³ give $\Psi(\mathbf{R}, r, \phi, \dots) = \exp(-i\chi)\Psi(\mathbf{R}, r, \phi - \pi, \dots)$. This in general makes for a many-valued wavefunction (similar to Riemann sheets for complex analytic functions).

Why can't we get a general χ in three dimensions?

(a) *Show, in three dimensions, that $\exp(i\chi) = \pm 1$, by arguing that a counter-clockwise rotation and a clockwise rotation must give the same phase.* (Hint: The phase change between ϕ and $\phi + \pi$ cannot change as we wiggle the path taken to swap the particles, unless the particles hit one another during the path. Try rotating the counter-clockwise path into the third dimension: can you smoothly change it to clockwise? What does that imply about $\exp(i\chi)$?)

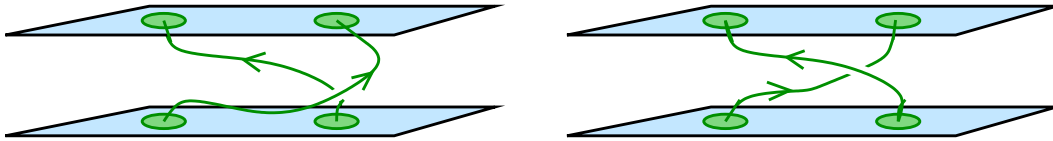


Fig. 3 Braiding of paths in two dimensions. In two dimensions, one can distinguish swapping clockwise from counter-clockwise. Particle statistics are determined by representations of the *Braid group*, rather than the permutation group.

Figure 3 illustrates how in two dimensions rotations by π and $-\pi$ are distinguishable; the trajectories form ‘braids’ that wrap around one another in different ways. You can’t change from a counter-clockwise braid to a clockwise braid without the braids crossing (and hence the particles colliding).

An angular boundary condition multiplying by a phase should seem familiar: it’s quite similar to that of the Bohm-Aharonov effect we studied in exercise 2.4. Indeed, we can implement fractional statistics by producing *composite particles*, by threading a magnetic flux tube of strength Φ through the center of each 2D boson, pointing out of the plane.

(b) *Remind yourself of the Bohm-Aharonov phase incurred by a particle of charge e encircling counter-clockwise a tube of magnetic flux Φ . If a composite particle of charge e and flux Φ encircles another identical composite particle, what will the net Bohm-Aharonov phase be?* (Hint: You can view the moving particle as being in a fixed magnetic field of all the other particles. The moving particle doesn’t feel¹⁴ its own

¹³The phase of the wave-function doesn’t have to be the same for the swapped particles, but the gradient of the phase of the wavefunction is a physical quantity, so it must be minus for the counter-clockwise path what it was for the clockwise path.

¹⁴Wilczek’s original paper, eqn (5), seems to suggest otherwise. I believe he assumes a contribution to the phase due to the flux tube encircling the electron: two jumps in phase due to the singular gauge transformation. I’m not absolutely sure why he’s wrong. But this is incompatible with more recent literature.

flux.)

(c) Argue that the phase change $\exp(i\chi)$ upon swapping two particles is exactly half that found when one particle encircles the other. How much flux is needed to turn a boson into an anyon with phase $\exp(i\chi)$? (Hint: The phase change can't depend upon the precise path, so long as it braids the same way. It's *homotopically invariant*, see chapter 9 of "Entropy, Order Parameters, and Complexity".)

Anyons are important in the quantum Hall effect. What is the quantum Hall effect? At low temperatures, a two dimensional electron gas in a perpendicular magnetic field exhibits a Hall conductance that is quantized, when the *filling fraction* ν (electrons per unit flux in units of Φ_0) passes near integer and rational values.

Approximate the quantum Hall system as a bunch of composite particles made up of electrons bound to flux tubes of strength Φ_0/ν . As a perturbation, we can imagine later relaxing the binding and allow the field to spread uniformly.¹⁵

(d) **Composite bosons and the integer quantum Hall effect.** At filling fraction $\nu = 1$ (the 'integer' quantum Hall state), what are the effective statistics of the composite particle? Does it make sense that the (ordinary) resistance in the quantum Hall state goes to zero?

- The excitations in the *fractional* quantum Hall effect are anyons with fractional charge. (The $\nu = 1/3$ state has excitations of charge $e/3$, like quarks, and their wavefunctions gain a phase $\exp(i\pi/3)$ when excitations are swapped.)
- It is conjectured that, at some filling fractions, the quasiparticles in the fractional quantum Hall effect have *non-abelian* statistics, which could become useful for quantum computation.
- The composite particle picture is a central tool both conceptually and in calculations for this field.

¹⁵This is not nearly as crazy as modeling metals and semiconductors as non-interacting electrons, and adding the electron interactions later. We do that all the time – 'electrons and holes' in solid-state physics, '1s, 2s, 2p' electrons in multi-electron atoms, all have obvious meanings only if we ignore the interactions. Both the composite particles and the non-interacting electron model are examples of how we use *adiabatic continuity* – you find a simple model you can solve, that can be related to the true model by turning on an interaction.