**Problem Set 13: "Second quantization", resonances, and supersymmetry**
**Graduate Quantum I**
**Physics 6572**
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**Potentially useful reading**
Sakurai and Napolitano, sections 5.5, 5.7 (time-dependent perturbation theory), 5.9 (decay
widths), 7.5 (second quantization)
Schumacher & Westmoreland section 14.6 (occupation numbers)
Weinberg sections 6.1-2 (time-dependent perturbation theory)


13.1 **Anticommutation and number.** ("Second quantization") ③

(Corrected version of Sakurai & Napolitano, problem 7.7.)

Suppose an operator $a$ and its adjoint $a^\dagger$ obey the fermion anticommutation relations
$\{a, a\} = \{a^\dagger, a^\dagger\} = 0$ and $\{a, a^\dagger\} = aa^\dagger + a^\dagger a = 1$. *Show that the only eigenvalues of*
*the operator $N = a^\dagger a$ are 0 and 1.*

13.2 **Phonons on a string.** (Quantum, Condensed matter) ③

A continuum string of length $L$ with mass per unit length $\mu$ under tension $\tau$ has a
vertical, transverse displacement $u(x, t)$. The kinetic energy density is $(\mu/2)(\partial u/\partial t)^2$
and the potential energy density is $(\tau/2)(\partial u/\partial x)^2$. The string has fixed boundary
conditions at $x = 0$ and $x = L$.

*Write the kinetic energy and the potential energy in new variables, changing from $u(x, t)$*
*to normal modes $q_k(t)$ with $u(x, t) = \sum_n q_{k_n}(t) \sin(k_n x)$, $k_n = n\pi/L$. Show in these vari-*
*ables that the system is a sum of decoupled harmonic oscillators. Calculate the density*
*of normal modes per unit frequency $g(\omega)$ for a long string $L$. Calculate the specific heat*
*of the string $c(T)$ per unit length in the limit $L \to \infty$, treating the oscillators quantum*
*mechanically.* (You can find the specific heat of one harmonic oscillator in section 7.2
of my book 'Entropy, Order Parameters, and Complexity'.) *What is the specific heat*
*of the classical string?* (Hint: The Hamiltonian is the *integral* of the energy density.)


Almost the same calculation, in three dimensions, gives the low-temperature specific
heat of crystals.

13.3 **Density Matrices and Statistical Mechanics.** (Quantum Stat Mech) ③

Quantum tunneling of atoms dominates the low temperature properties of glasses (as discovered at Cornell by Robert Pohl and his Master's student Zeller). Defects in crystals also have important quantum tunneling properties; indeed, tunneling defects in alkali halides were a major field of study here in the 60's and 70's (Pohl, Sievers, Silsbee, Krumhansl, ...). For example, if you substitute a (small) lithium atom for a (larger) potassium in KCl, it lowers its energy by sitting off-center, nestled into a corner of the cube formed by its six Cl neighbors. But quantum mechanically, it has six such off-center positions, and can tunnel between them.

Here we'll study the simpler case of an atom with two equilibrium positions. Let the Hamiltonian for an atom in a symmetric double well be approximated by

$$\mathcal{H}_0 = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix} \tag{1}$$

where the basis states $|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are localized in the left and right wells, and where $\Delta > 0$ is the tunneling matrix element (calculated, for example, by WKB or instantons). This two-level system approximation is the starting point for many theories of defect tunneling and glasses.

*(a) Find the eigenvalues and eigenvectors of $\mathcal{H}_0$.*

At a temperature $T$, quantum statistical mechanics tells us that the density matrix is

$$\rho = \exp(-\mathcal{H}/k_B T)/\mathrm{Tr}(\exp(-\mathcal{H}/k_B T)) \tag{2}$$

*(b) Find the density matrix $\rho(T)$ for $\mathcal{H}_0$.* (Hint: There are lots of different ways to exponentiate the matrix. Your final answer should be written without any infinite sums, and the diagonal elements should make sense.)

*(c) Find the expectation value for the energy $U(T) = \langle H \rangle$ by taking an appropriate trace involving $\rho$. Find the specific heat $c(T) = \mathrm{d}U/\mathrm{d}T$.*

Your formula should give a peak in the specific heat near $k_B T = \Delta$. This is called a Schottky peak, and is often a clear signal of a tunneling defect.

Statistical mechanics is often formulated in the energy basis. Every energy eigenstate $|E_\alpha\rangle$ is weighted by a Boltzmann factor $\exp(-E_\alpha/k_B T)$, so the probability of being in that state is $p_\alpha = \exp(-E_\alpha/k_B T)/Z$.

$$\rho = \sum_\alpha p_\alpha |E_\alpha\rangle\langle E_\alpha|. \tag{3}$$

Here the partition function $Z = \sum_\alpha \exp(-E_\alpha/k_B T)$ is seen as the normalization factor for the Boltzmann sum.

*(d) Calculate the expectation value for the energy by summing over the eigenstates. Check your answer from part (c).*

Why bother with density matrices, when eigenstates will do? In many cases, the eigenstate basis isn't natural. For example, when our double-well atom is put in an electric field or under strain, the couplings are simple in the position basis, and quite ugly and unnatural in the energy eigenstates. For example, if the left and right wells are separated by a distance $Q$ and the ion has charge $e$, the total Hamiltonian might be $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$, with an interaction Hamiltonian

$$\mathcal{H}_I = eEX = eE \begin{pmatrix} -Q/2 & 0 \\ 0 & Q/2 \end{pmatrix}. \tag{4}$$

*(d) Write $\mathcal{H}_I$ in the energy basis of the unperturbed Hamiltonian.* (Hint: The answer isn't so messy, but only because it's a symmetric double well.)

### 13.4 **Resonances: $\alpha$-decay.** (Quantum) ③
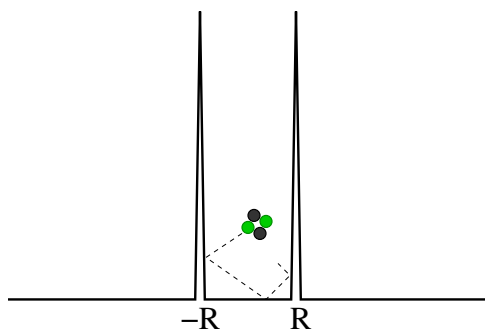


**Fig. 1** *One-dimensional nuclear potential.*

In this exercise, we solve a one-dimensional model of radioactive $\alpha$-decay, where a nucleus ejects a particle formed by two protons and two neutrons (a Helium-4 nucleus).

We assume that the strong force minus the Coulomb repulsion provides a constant potential for the $\alpha$ particle inside a nucleus of radius $R$, which for simplicity we shall assume is zero. At the edge of the nucleus in the real world, the (short-range) strong interaction drops rapidly to zero, but the Coulomb repulsion decays slowly with distance, leading to a tunneling barrier. We model this barrier with a $\delta$-function of strength $U > 0$[1] (see Fig. 1). Both inside and outside the nucleus, the potential is zero:

$$V(x) = U\delta(x \pm R)$$

(The attractive case $U < 0$ is a model for the hydrogen molecule, and is discussed for example in Wikipedia's *Double Delta Potential* article, http://en.wikipedia.org/wiki/Delta_potential#Double_Delta_Potential.)

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[1] In one-dimensional quantum mechanics, the first derivative of the wave-function jumps where the potential has a $\delta$-function. Find details in a textbook or on the Web.

Parts (a)-(c) of this exercise solve analytically for the energy eigenstates, but getting them correct is important for the later parts.[2]

Our Hamiltonian has a symmetry which allows us to choose energy eigenstates that are even ($\psi_E$) or odd ($\phi_E$).

(a) *What symmetry of the Hamiltonian is this? Given an energy eigenstate $\zeta_E(x)$ with mixed symmetry (in particular, $\zeta_E$ is not odd), construct an even eigenstate of the same energy (ignoring the overall normalization).*

In this exercise, we will be interested in the family of even eigenstates states $\psi_E$ which can be non-zero at $x = 0$, and for which $\psi'_E(0) = \partial \psi_E / \partial x |_{x=0} = 0$. To solve for these even energy eigenstates, there are three steps.

First, we deduce the form of the wavefunction. Note that, away from the $\delta$-function, the wavefunction has wave-vector $k(E) = \sqrt{2mE}/\hbar$; it is convenient to label the wave-functions by $k(E)$ instead of $E$. Using the boundary condition at zero, we write the wavefunction for $|x| < R$ as $\psi_k^{\text{nuc}} = A_k \cos(kx)$, with an overall amplitude $A_k$. For $x > R$, we write the wavefunction as a standing sine wave[3] $\psi_k^{\text{out}} = B \sin(kx + \Delta_k)$. Note that there is a continuum of $\psi_k$ eigenstates, so it is proper for us to use the $\delta$-function normalization $\langle \psi_k | \psi_{k'} \rangle = \delta(k - k')$.

(b) *Show that $B = 1/\sqrt{\pi}$ for our continuum wavefunction to be properly normalized. (Hints: Since we're studying only even eigenstates, $k \geq 0$. Also, because the region $|x| < R$ is finite, we can ignore it for the normalization in an infinite box.)*

Second, we impose the conditions induced by the $\delta$-potential at the edge of the nucleus.

(c) *Write the condition on $A_k$ and $\Delta_k$ given by imposing continuity of $\psi_k(x)$ at $x = R$. Write the conditions on $A_k$ and $\Delta_k$ given by the discontinuity of $\psi'_k(x)$ imposed by the $\delta$-function potential (see footnote 1). For convenience, write your answers from here on in terms of the unitless ratio $\widetilde{U} = 2mRU/\hbar^2$.*

Third, we solve for the eigenstates of our Hamiltonian that are non-zero at $x = 0$.

(d) *Use the conditions of part (c), solve for $A_k^2$. (Trick: Arrange the two equations of part (c) to be $\sin(kR + \Delta_k) = \cdots$ and $\cos(kR + \Delta_k) = \cdots$, where $\cdots$ is independent of $\Delta_k$. Sum the squares of the right-hand sides: what must the sum be equal to?)*

We now consider the decay of an $\alpha$-particle injected into this potential at $x = 0$. That is, consider an initial wavefunction $\Psi(x) = \delta(x)$.[4]

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[2]Feel free to check your answers by solving Schrödinger's equation numerically, approximating $\delta(x - R) = (1/\sqrt{2\pi\sigma^2}) \exp(-x^2/(2\sigma^2))$ for $\sigma$ as small as is numerically convenient.

[3]For $x < -R$, we use the even symmetry of $\psi_E$ to set $\psi_k = \psi_k^{\text{out}}(-x) = B \sin(-kx + \Delta_k)$. Note that we are solving for standing waves in this problem. For other purposes, scattering waves or outgoing waves might be preferable.

[4]This is a nuclear version of tunneling from an STM tip; $P(E) = P(k(E)) \, (dk/dE)$ measures the local density of states for the $\alpha$ particle at the center of the nucleus.

(e) *What is the probability[5] $P(k)$ of being in eigenstate $\psi_k$?* (Write your answer abstractly in terms of $\psi_k(x)$. This you can do without solving parts (a-d).)

(f) *Plot the probabilities $P(k)$ versus $kR$ with $\widetilde{U} = 30$ and for $0 < kR < 10$.*

In the limit $U \to \infty$, the nucleus should approximate a particle in a box of size $2R$. In that limit, the injection of an $\alpha$-particle can only occur at certain discrete energies – the nuclear eigenstates $\mathcal{E}_m^\infty$ of a free particle in a box of size $2R$.

(g) *Compare the peaks you found in part (f) to the wavevectors for the particle-in-a-box states. Why are you missing half of the peaks?*

(h) *(Extra credit) Change variables from $P(k)$ to $P(E)$ by using $\mathrm{d}E/\mathrm{d}k$.[6] Using the FWHM of the peaks in $P(E)$, estimate the lifetimes of the first three even resonances of our nucleus (either numerically or analytically). Calculate the integrated probability for being in each of these three resonances. Do they go to the 'particle-in-a-box' values as $U \to \infty$?*

13.5 **Supersymmetric harmonic oscillator.**[7] (Quantum) ③

One of the main predictions of supersymmetry[8] is that each particle comes with a supersymmetric partner with the same mass but with opposite statistics.[9] For example, the fermionic electron is paired with the bosonic selectron. Supersymmetry is also a potential symmetry of nature, with an unusual connection to the translational symmetries in space and time (the Poincaré group). Finally, supersymmetry allows one to calculate remarkable things about certain Hamiltonians. In this exercise, we shall explore a "zero-dimensional"[10] example of a supersymmetric Hamiltonian, and try to illustrate each of these features of supersymmetry.[11]

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[5]The position eigenstate $\Psi(x) = |x = 0\rangle$ is $\delta$-function normalized, with $\langle x|x'\rangle = \delta(x - x')$. Hence the 'probability' $P(k)$ integrates to infinity, and not to one. You can alternatively think of this calculation as the first step in evaluating the Green's function $G(x', t'; 0, 0)$ from $x = t = 0$, which evolves an initial packet $\delta(x)$ from the origin.

[6]The overall normalization of these densities of states may be off by a factor with dimensions of length. At root, this is because I started with a $\delta$-function wave packet, whose squared norm is infinity rather than one. I should have put the system on a lattice and taken the lattice to zero, or used Green's functions. Feel free to proceed.

[7]Developed in collaboration with John Stout, Fall 2013.

[8]The footnotes in this problem are meant as inspiration – tying it to fundamental ideas in theoretical physics. *None of the footnotes are necessary or useful for solving the problem* – ignore them if you wish.

[9]Supersymmetric partners have the same mass as long as supersymmetry is unbroken. We expect supersymmetry to be *spontaneously broken* at low energy scales, given that we have not yet detected any supersymmetric partners of the Standard Model particles.

[10]We often talk about quantum field theories in $d$ spatial dimensions and one time dimension as $d+1$-dimensional field theories: our space-time is thus 3+1 dimensional. We can view non-relativistic quantum mechanics as a $d = 0$ quantum field theory, and it is in this regard that we consider the supersymmetric Hamiltonians described here as "zero-dimensional" or 0+1-dimensional.

[11]There are a number of discussions of the supersymmetric harmonic oscillator and zero-dimensional supersymmetry in the literature and on the Web. Feel free to consult these. If you find one particularly useful, reference it properly in your writeup.

Remember the commutation relations for creation and annihilation operators suitable for bosons

$$[a, a^\dagger] = 1 \qquad [a, a] = [a^\dagger, a^\dagger] = 0, \tag{5}$$

and fermions

$$\{b, b^\dagger\} = 1 \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0. \tag{6}$$

where $[A, B] = AB - BA$ is the commutator and $\{A, B\} = AB + BA$ is the anticommutator.[12] For this simple example, we take our bosons and fermions to be noninteracting, so their creation and annihilation operators commute,

$$[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0. \tag{7}$$

In one dimension, the Hamiltonian of the simple harmonic oscillator of frequency $\omega$ can be written either in terms of $x$ and $p$:

$$\mathcal{H}_B = p^2/2m + \tfrac{1}{2}m\omega^2 x^2 \tag{8}$$

or in terms of the creation and annihilation operators

$$\mathcal{H}_B = \hbar\omega(a^\dagger a + \tfrac{1}{2}). \tag{9}$$

Here $\tfrac{1}{2}\hbar\omega$ is the ground state energy of the harmonic oscillator – the zero-dimensional analogue of the 'vacuum energy' in field theory.

The harmonic oscillator Hamiltonian can be written in a more symmetric way by using the anticommutator.

(a) *Show that* $\mathcal{H}_B = \tfrac{1}{2}\hbar\omega\{a^\dagger, a\}$. *Is the vacuum energy still* $\tfrac{1}{2}\hbar\omega$?

Note that we're now calling the ladder operators $a$ and $a^\dagger$ 'creation' and 'annihilation' operators. In this new language, the $n^{th}$ excited state of the harmonic oscillator can be viewed as a state with $n$ bosons.

Define a 'fermionic harmonic oscillator' in analogy to the bosonic one, $\mathcal{H}_F = \tfrac{1}{2}\hbar\omega[b^\dagger, b]$. Again, we can view the $n^{th}$ excited state as a state of $n$ fermions.

(b) *What is the ground state energy of* $\mathcal{H}_F$? *How many fermions are in the ground state, in this new language? What is the energy of the state with one fermion?*

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[12]Be sure to avoid getting confused by our multiple uses of the terms 'boson' and 'fermion' in this exercise. There are really three different ways we use the terms, each extremely useful and compelling. They are:

(a) the objects which vibrate or have spins, that produce harmonic oscillators or two-state systems,

(b) the 'primitive' bosons (and fermions) which are excitations within a harmonic oscillator (e.g., N bosons = Nth excited state inside the vibrating object)

(c) the composite objects inside the supersymmetric Hamiltonian that merge zero or more 'primitive' bosons and fermions.

(c) *If we write the zero-fermion state as*[13] $|0\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ *and the one-fermion state as* $|1\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, *then write $b$, $b^\dagger$, and $\mathcal{H}_F$ in terms of the three Pauli matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$. Check that your form for $b$ and $b^\dagger$ satisfy the anticommutation relations of eqn 6* (Remember $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.)

We can write our first supersymmetric Hamiltonian by adding the boson and fermion harmonic oscillators:

$$\mathcal{H}_S = \mathcal{H}_B + \mathcal{H}_F = \tfrac{1}{2}\hbar\omega \left( \{a^\dagger, a\} + [b^\dagger, b] \right). \tag{10}$$

Note that the ground state energy for this Hamiltonian is zero.[14]

This supersymmetric Hamiltonian is not particularly difficult to solve. Because there is no interaction between the bosonic and fermionic parts of the Hamiltonian, the solution separates and the eigenstates are just products $\psi(x)\chi(s)$, and the energy of the eigenstate is the sum of the Fermi and Bose energies.

Remember that a composite particle with an odd number of primitive fermions is a fermion – so half of our eigenstates represent composite bosons, and half represent composite fermions.

(d) *Solve for the energies for the eigenstates of $\mathcal{H}_S$. Which eigenstates represent composite fermions? Which composite bosons? Draw the 'level diagram' for $\mathcal{H}_S$, with the first few composite boson eigenenergies as a column of horizontal lines on the left, and the first few composite fermion eigenenergies on the right. On each line, write the number of primitive bosons and fermions making up the composite. Is there a composite fermion state for each composite boson state? What state is the exception?* We shall hitherto drop the 'composite' label. If we interpret the energy of a state as the mass of a particle[15] , supersymmetry gives us for every fermion a boson with the same mass.

The fact that our Hamiltonian has (almost) one fermion state for each boson state is a result of an unusual symmetry of the Hamiltonian. To see this, let's define an operator, called the *supercharge*,

$$Q = b \left( \frac{p}{\sqrt{m}} + \mathrm{i}\sqrt{m}\omega x \right) = \mathrm{i}\sqrt{2\hbar\omega}\, b a^\dagger. \tag{11}$$

(Remember that $x = \sqrt{\hbar/2m\omega}(a^\dagger + a)$ and $p = \mathrm{i}\sqrt{m\omega\hbar/2}(a^\dagger - a)$.)

(e) *Show that $[\mathcal{H}_S, Q] = 0$. (Hence $Q$ is a symmetry of the Hamiltonian.) Show that $Q$ acting on a fermion state gives a constant times a boson state of the same energy, and*

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[13] I apologize again for the shift back in notation. In this problem, we revert back to the notation used in lecture: $|1\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, instead of the quantum computing notation used in an earlier exercise.

[14] That is a hint for part (b).

[15] We can motivate this by remembering that we are dealing with a theory with zero spatial dimensions, and so the usual relativistic energy of a particle (which should correspond to a eigenstate of our Hamiltonian) $E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$ reduces to $E = mc^2$. We often interpret the mass of a particle as being the energy required to create a "copy" of the particle at rest, and it is analogous to the band gap energy in semiconductors.

*that $Q^\dagger$ acting on a boson state almost always gives a constant times a fermion state of the same energy. Which of the ground states is the exception to this rule? Show that this ground state is an eigenfunction of $Q$ and $Q^\dagger$ with eigenvalue zero.*[16]

Supersymmetry has been shown (by Haag, Lopuszanski, and Sohnius[17]) to be the only way to consistently extend the symmetries of spacetime. Spacetime has a spatial translational symmetry (with an associated conserved momentum), a time-translational symmetry (associated with the conserved energy, with the Hamiltonian giving the infinitesimal time-translation operator), and other symmetries (rotations and relativistic boosts). Combining these symmetries gives us the Poincaré group.

In our "zero-dimensional" harmonic oscillator, only the time-translational symmetry remains from the Poincaré group. How does supersymmetry extend time-translation invariance? Can we somehow create a time translation by supersymmetrically transforming it?

(f) *Show that $\mathcal{H}_S = \tfrac{1}{2}\{Q, Q^\dagger\}$.* We see that a combination of two supercharges generates a time translation!

The supersymmetric harmonic oscillator we looked at above may seem pretty trivial: how hard is it to get degenerate states when all states have the same energy splitting?

However, we can generate lots of interacting supersymmetric Hamiltonians by specifying a supercharge

$$Q_W = b\left(\frac{p}{\sqrt{m}} + \mathrm{i}\sqrt{m}W'(x)\right) \tag{12}$$

where $W'(x) = \mathrm{d}W/\mathrm{d}x$, and requiring that $\mathcal{H}_W = \tfrac{1}{2}\{Q_W, Q_W^\dagger\}$, where the real function $W(x)$ is called the *superpotential*.

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[16]$Q\Psi = 0$ gives us a first-order differential equation which can be directly integrated to obtain this ground state wave function! This trick extends to field theory applications too – yet another way in which supersymmetry simplifies theorists lives.

[17]The story starts with the Coleman-Mandula no-go theorem in 1967. (According to n-Lab, a no-go theorem is "any theorem...that shows that an idea is not possible even though it may appear as if it should be." Thus Bell's theorem is a no-go theorem dictating the impossibility of local, hidden variable theories that reproduce the predictions of quantum mechanics.) The Coleman-Mandula theorem tells us that in a realistic quantum field theory, space-time symmetries (like the Lorentz group) can only be combined with internal symmetries (like the SU(3) of the strong interaction) in a trivial way (so that the total symmetry group is (space-time symmetry)×(internal symmetry group).

How did the no-go theorem go? You may remember, according to Noether's theorem, that all continuous symmetries are associated with conserved quantities: thus momentum and energy are the conserved quantities related to translations in space and time, and conversely **p** and $\mathcal{H}$ (or $P^j$ and $P^0$ in four-vector notation) generate infinitesimal space and time translations. Coleman and Mandula showed that spacetime symmetry generators had to commute with generators of any new internal symmetries represented by commutation relations.

Haag, Lopuszanski, and Sohnius were able to skirt the Coleman-Mandula theorem by avoiding the hidden assumption that the new symmetry had to obey commutation relations: the new supersymmetries involve *anticommutation* relations. In fact, they were able to show that this is the *only* way of extending the Poincaré group for consistent, interacting quantum field theories with massive particles.

Our Hamiltonian $\mathcal{H}_S$ can be viewed as the special case of $W(x) = \frac{1}{2}\omega x^2$. Note that our superpotential need not have units of energy.

(g) *Show that $Q_W$ and $\mathcal{H}_W$ as 2×2 matrices*

$$\mathcal{H}_W = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix} \quad and \quad Q_W = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}. \tag{13}$$

*where the elements of the matrices are functions of $p$ and $x$.* (Hint: Remember $p = -i\hbar\partial/\partial x$. You might check this against the Web, which has different units.)

There is a lovely relationship between the eigenvalues and eigenfunctions of $\mathcal{H}_1$ and $\mathcal{H}_2$, two seemingly different Hamiltonians. Let $\Psi_n^{(1)}(x)$ and $\Psi_m^{(2)}(x)$ be the $n$-th and $m$-th eigenfunctions of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(h) *Using the fact that $[\mathcal{H}_W, Q_W] = [\mathcal{H}_W, Q_W^\dagger] = 0$, show that $A^\dagger \Psi_m^{(2)}(x)$ is an eigenstate of $\mathcal{H}_1$ and $A\Psi_n^{(1)}(x)$ is an eigenstate of $\mathcal{H}_2$.* (Thus, if we know the eigenfunctions and eigenenergies of one of the Hamiltonians, we know them for the other.)

Let us work out a specific example. Consider $W'(x) = (\pi\hbar/mL)\cot(\pi x/L)$.

(i) *Show that $\mathcal{H}_1$ is the particle-in-a-box Hamiltonian (Fig. 2) shifted by a constant to set its ground state energy to zero. Show that $\mathcal{H}_2$ is a Hamiltonian with potential*[18]

$$V(x) = \frac{\pi^2\hbar^2}{2mL^2}\left(2\csc^2\left(\frac{\pi x}{L}\right) - 1\right). \tag{14}$$

*Using the first excited state $\Psi_2^{(1)}(x) = \sqrt{2/L}\sin(2\pi x/L)$ of $\mathcal{H}_1$ and the operator $A$, generate the ground state of $\mathcal{H}_2$ and show that it is proportional to $\sin^2(\pi x/L)$. Explicitly show (taking the derivatives) that $A\Psi_2^{(1)}(x)$ is an eigenfunction of $\mathcal{H}_2$ and thus verify that its energy is the same as that of $\Psi_2^{(1)}$.*
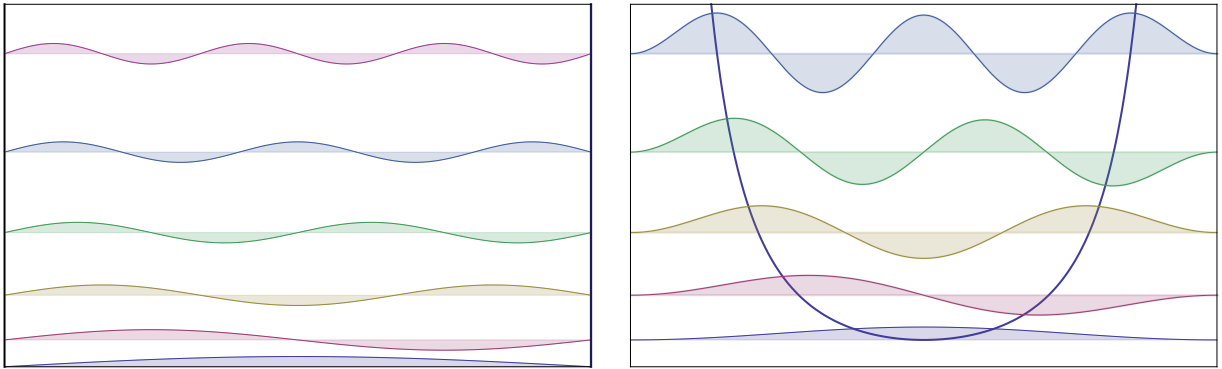


**Fig. 2 Supersymmetric eigenenergies and eigenstates**. (Left) Eigenstates for $\mathcal{H}_1$, the square well potential, displaced vertically by their eigenenergies. (Right) Eigenstates for $\mathcal{H}_2$, the $\csc^2 x$ potential, which is the supersymmetric pair for the square well.

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[18]Note that the $\Psi = 0$ boundary conditions for the two Hamiltonians are the same for both $\mathcal{H}_1$ and $\mathcal{H}_2$.

While supersymmetry may not exist in nature, it has proved to be an excellent tool for gaining insight into the way theories with gauge symmetry behave. (For example, we have no proof that the strong interaction confines quarks, but Seiberg and Witten were able to demonstrate confinement in certain supersymmetric theories.) It also has allowed physicists to prove theorems in pure mathematics. Ed Witten, high-energy theorist at the Institute for Advanced Study, was awarded the Fields Medal (the Nobel equivalent in math) for his use of supersymmetry to figure out topological properties of a manifold (such as the Euler characteristic, related to the number of holes or handles a manifold has) by using the difference in the number of zero-energy 'fermion' and 'boson' wavefunctions on it.[19]

[19]E. Witten, *Supersymmetry and Morse Theory*, J. Diff Geom. **17**, 661692 (1982).