Problem Set 4: Bosons, Fermions, and Anyons
Graduate Quantum I
Physics 6572

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Reading

Sakurai and Napolitano, sections 3.1-3.3, chapter 4, and chapter 7
Sethna, "Entropy, Order Parameters, and Complexity", ch. 7.2, 7.3, 7.6
Readings from individual exercises

4.1 **Rotating Fermions.** (Group theory) ③

In this exercise, we'll explore the *geometry* of the space of rotations.

Spin 1/2 fermions transform upon rotations under $SU(2)$, the unitary 2×2 matrices with determinant one. Vectors transform under $SO(3)$, the ordinary 3×3 rotation matrices you know of.

Sakurai argues that a general $SU(2)$ matrix $U = \left(\begin{smallmatrix} a & b \\ -b^* & a^* \end{smallmatrix}\right)$ with $|a|^2 + |b|^2 = 1$. Viewing $\{\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)\}$ as a vector in four dimensions, $SU(2)$ then geometrically is the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$.

Remember that for every matrix R in $SO(3)$, there are two unitary matrices U and $-U$ corresponding to the same physical rotation. The matrix $-U$ has coordinates $(-a, -b)$ – it is the antipodal point on $\mathbb{S}^3$, exactly on the opposite side of the sphere. So $SO(3)$ is geometrically the unit sphere with *antipodal points identified*. This is called (for obscure reasons) the *projective plane*, $\mathbb{R}P^3$.

Feynman's plate (in Feynman's plate trick) as it rotates 360° travels in rotation space from one orientation to its antipode. While I'm not sure anyone has figured out whether arms, shoulders, and elbows duplicate the properties of fermions under rotations, the plate motion illustrates the possibility of a minus sign.

But we can calculate this trajectory rather neatly by mapping the rotations not to the unit sphere, but to the space $\mathbb{R}^3$ of three-dimensional vectors. (Just as the 2-sphere $\mathbb{S}^2$ can be projected onto the plane, with the north pole going to infinity, so can the 3-sphere $\mathbb{S}^3$ be projected onto $\mathbb{R}^3$.) Remember the axis-angle variables, where a rotation of angle ϕ about an axis $\hat{\mathbf{n}}$ is given by

$$\exp(-\mathrm{i}\mathbf{S} \cdot \hat{\mathbf{n}}\phi/\hbar) = \exp(-\mathrm{i}\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi/2) = \exp(-\mathrm{i}\mathbf{J} \cdot \hat{\mathbf{n}}\phi/\hbar) \tag{1}$$

where the middle formula works for $SU(2)$ (where $\mathbf{S} = \hbar\sigma/2$, because the particles have spin 1/2) and the last formula is appropriate for $SO(3)$.[1] We figure $\mathbf{n}$ will be the

[1]The factor of two is there for $SU(2)$ because the spin is 1/2. For $SO(3)$, infinitesimal generators are antisymmetric matrices, so $\mathbf{S}^{(i)}_{jk} = \mathbf{J}^{(i)}_{jk} = \mathrm{i}\hbar\epsilon_{ijk}$ in the xyz basis; in the usual quantum basis $m_z = (-1, 0, 1)$ the formula will be different.

direction of the vector in $\mathbb{R}^3$ corresponding to the rotation, but how will the length depend on ϕ? Since all rotations by 360° are the same, it makes sense to make the length of the vector go to infinity as $\phi \to 360°$. We thus define the *Modified Rodrigues coordinates* for a rotation to be the vector $\mathbf{p} = \hat{\mathbf{n}} \tan(\phi/4)$.

(a) *Fermions, when rotated by 360°, develop a phase change of* $\exp(\mathrm{i}\pi) = -1$ *(as discussed in Sakurai & Napolitano p. 165, and as we illustrated with Feynman's plate trick). Give the trajectory of the modified Rodrigues coordinate for the fermion's rotation as the plate is rotated* $720°$ *about the axis* $\hat{\mathbf{n}} = \hat{z}$. (We want the continuous trajectory on the sphere $\mathbb{S}^3$, perhaps which passes through the point at ∞. Hint: The trajectory is already defined by the modified Rodrigues coordinate: just describe it.)

(b) *For a general Rodrigues point* $\mathbf{p}$ *parameterizing a rotation in* $SO(3)$*, what antipodal point* $\mathbf{p}'$ *corresponds to the same rotation?* (Hint: A rotation by ϕ and a rotation by $\phi + 2\pi$ should be identified.)

4.2 **Quantum dice.**[2] (Quantum) ②

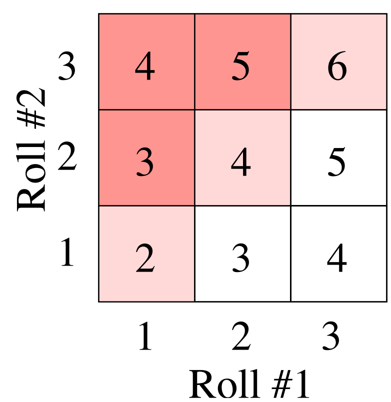

Fig. 1 Quantum dice. Rolling two dice. In *Bosons*, one accepts only the rolls in the shaded squares, with equal probability 1/6. In *Fermions*, one accepts only the rolls in the darkly-shaded squares (not including the diagonal from lower left to upper right), with probability 1/3.

You are given several unusual 'three-sided' dice which, when rolled, show either one, two, or three spots. There are three games played with these dice: *Distinguishable*, *Bosons*, and *Fermions*. In each turn in these games, the player rolls one die at a time, starting over if required by the rules, until a legal combination occurs. In *Distinguishable*, all rolls are legal. In *Bosons*, a roll is legal only if the new number is larger or equal to the preceding number. In *Fermions*, a roll is legal only if the new number is strictly larger than the preceding number. See Fig. 1 for a table of possibilities after rolling two dice.

[2] This exercise was developed in collaboration with Sarah Shandera.

Our dice rules are the same ones that govern the quantum statistics of identical particles.

(a) *Presume the dice are fair: each of the three numbers of dots shows up* $1/3$ *of the time. For a legal turn rolling a die twice in Bosons, what is the probability* $\rho(4)$ *of rolling a 4? Similarly, among the legal Fermion turns rolling two dice, what is the probability* $\rho(4)$*?*

(b) *For a legal turn rolling three 'three-sided' dice in Fermions, what is the probability* $\rho(6)$ *of rolling a 6?* (Hint: There is a Fermi exclusion principle: when playing *Fermions*, no two dice can have the same number of dots showing.) Electrons are fermions; no two electrons can be in exactly the same state.

When rolling two dice in *Bosons*, there are six different legal turns (11), (12), (13), ..., (33); half of them are doubles (both numbers equal), while for plain old *Distinguishable* turns only one-third would be doubles[3]; the probability of getting doubles is enhanced by 1.5 times in two-roll *Bosons*. When rolling three dice in *Bosons*, there are ten different legal turns (111), (112), (113), ..., (333). When rolling M dice each with N sides in *Bosons*, one can show that there are

$$\binom{N+M-1}{M} = \frac{(N+M-1)!}{M!\,(N-1)!}$$

legal turns.

(c) *In a turn of three rolls, what is the factor by which the probability of getting triples in Bosons is enhanced over that in Distinguishable? In a turn of* M *rolls, what is the enhancement factor for generating an M-tuple (all rolls having the same number of dots showing)?*

Notice that the states of the dice tend to cluster together in *Bosons*. Examples of real bosons clustering into the same state include Bose condensation and lasers (Exercise 3).

4.3 **Bosons are gregarious: superfluids and lasers.** (Quantum, Optics, Atomic physics) ③

Adding a particle to a Bose condensate. Suppose we have a non-interacting system of bosonic atoms in a box with single-particle eigenstates ψ_n. Suppose the system begins with all N bosons in a state ψ_0 (a "Bose condensed state"), so

$$\Psi_N^{[0]}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \psi_0(\mathbf{r}_1) \cdots \psi_0(\mathbf{r}_N). \tag{2}$$

[3] For *Fermions*, of course, there are no doubles.

Suppose a new particle is gently injected into the system, into an equal superposition of the M lowest single-particle states.[4] That is, if it were injected into an empty box, it would start in state

$$\phi(\mathbf{r}_{N+1}) = \frac{1}{\sqrt{M}}\big(\psi_0(\mathbf{r}_{N+1}) + \psi_1(\mathbf{r}_{N+1}) \\ + \ldots + \psi_{M-1}(\mathbf{r}_{N+1})\big). \tag{3}$$

The state $\Phi(\mathbf{r}_1, \ldots \mathbf{r}_{N+1})$ after the particle is inserted into the non-interacting Bose condensate is given by symmetrizing the product function $\Psi_N^{[0]}(\mathbf{r}_1, \ldots, \mathbf{r}_N)\phi(\mathbf{r}_{N+1})$

$$\Psi_{\text{sym}}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = (\text{normalization}) \sum_P \Phi(\mathbf{r}_{P_1}, \mathbf{r}_{P_2}, \ldots, \mathbf{r}_{P_N}). \tag{4}$$

(a) *Calculate the symmetrized initial state of the system with the injected particle. Show that the ratio of the probability that the new boson enters the ground state ψ_0 is enhanced over that of its entering a particular empty state[5] (ψ_m for $0 < m < M$) by a factor $N + 1$.* (Hint: First do it for $N = 1$.)

So, if a macroscopic number of bosons are in one single-particle eigenstate, a new particle will be much more likely to add itself to this state than to any of the microscopically populated states.

Notice that nothing in your analysis depended on ψ_0 being the lowest energy state. If we started with a macroscopic number of particles in a single-particle state with wavevector $\mathbf{k}$ (that is, a superfluid with a supercurrent in direction $\mathbf{k}$), new added particles, or particles scattered by inhomogeneities, will preferentially enter into that state. This is an alternative approach to understanding the persistence of supercurrents, complementary to the topological approach (Exercise 2.5).

Adding a photon to a laser beam. This 'chummy' behavior between bosons is also the principle behind lasers.[6] A laser has N photons in a particular mode. An atom in an excited state emits a photon. The photon it emits will prefer to join the laser beam than to go off into one of its other available modes by a factor $N+1$. Here the N represents *stimulated emission*, where the existing electromagnetic field pulls out the energy from the excited atom, and the $+1$ represents *spontaneous emission* which occurs even in the absence of existing photons.

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[4]For free particles in a cubical box of volume V, injecting a particle at the origin $\phi(\mathbf{r}) = \delta(\mathbf{r})$ would be a superposition of *all* plane-wave states of equal weight, $\delta(\mathbf{r}) = (1/V)\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$. (In second-quantized notation, $a^\dagger(\mathbf{x} = 0) = (1/V)\sum_{\mathbf{k}} a^\dagger_{\mathbf{k}}$.) We 'gently' add a particle at the origin by restricting this sum to low-energy states. This is how quantum tunneling into condensed states (say, in Josephson junctions or scanning tunneling microscopes) is usually modeled.

[5]More precisely, calculate the ratio of the probability of being in the many-body ground state (all particles in state ψ_0) to the probability of injecting into the many-body state with one electron in the state ψ_m and the rest in ψ_0.

[6]Laser is an acronym for 'light amplification by the stimulated emission of radiation'.

Imagine a single atom in a state with excitation energy energy E and decay rate Γ, in a cubical box of volume V with periodic boundary conditions for the photons. By the energy-time uncertainty principle, $\langle \Delta E \, \Delta t \rangle \geq \hbar/2$, the energy of the atom will be uncertain by an amount $\Delta E \propto \hbar\Gamma$. Assume for simplicity that, in a cubical box without pre-existing photons, the atom would decay at an equal rate into any mode in the range $E - \hbar\Gamma/2 < \hbar\omega < E + \hbar\Gamma/2$.

(b) *Assuming a large box and a small decay rate* Γ, *find a formula for the number of modes* M *per unit volume* V *per unit energy* E *in the box (the density of states). How many states are competing for the photon emitted from our atom, for a laser with wavelength* $\lambda = 619\,\mathrm{nm}$ *and line-width* $\Gamma = 10^4\,\mathrm{rad/s}$. (Hint: The eigenstates are plane waves, with two polarizations per wavevector. Using periodic boundary conditions, one can derive the density of states. This is a standard calculation, so you can look up the answer to check it.)

Assume the laser is already in operation, so there are N photons in the volume V of the lasing material, all in one plane-wave state (a *single-mode* laser).

(c) *Using your result from part (a), give a formula for the number of photons per unit volume* N/V *there must be in the lasing mode for the atom to have 50% likelihood of emitting into that mode.*

The main task in setting up a laser is providing a population of excited atoms. Amplification can occur if there is a *population inversion*, where the number of excited atoms is larger than the number of atoms in the lower energy state (definitely a non-equilibrium condition). This is made possible by *pumping* atoms into the excited state by using one or two other single-particle eigenstates.

4.4 **Phonons on a string.** (Quantum, Condensed matter) ③

A continuum string of length L with mass per unit length μ under tension τ has a vertical, transverse displacement $u(x,t)$. The kinetic energy density is $(\mu/2)(\partial u/\partial t)^2$ and the potential energy density is $(\tau/2)(\partial u/\partial x)^2$. The string has fixed boundary conditions at $x = 0$ and $x = L$.

Write the kinetic energy and the potential energy in new variables, changing from $u(x,t)$ *to normal modes* $q_k(t)$ *with* $u(x,t) = \sum_n q_{k_n}(t) \sin(k_n x)$, $k_n = n\pi/L$. *Show in these variables that the system is a sum of decoupled harmonic oscillators. Calculate the density of normal modes per unit frequency* $g(\omega)$ *for a long string* L. *Calculate the specific heat of the string* $c(T)$ *per unit length in the limit* $L \to \infty$, *treating the oscillators quantum mechanically.* (You can find the specific heat of one harmonic oscillator in section 7.2 of my book 'Entropy, Order Parameters, and Complexity'.) *What is the specific heat of the classical string?* (Hint: The Hamiltonian is the *integral* of the energy density.)

Almost the same calculation, in three dimensions, gives the low-temperature specific heat of crystals.

4.5 **Anyons.** (Statistics) ③

Frank Wilczek, "Quantum mechanics of fractional-spin particles", *Phys. Rev. Lett.* **49**, 957 (1982).
Steven Kivelson, Dung-Hai Lee, and Shou-Cheng Zhang, "Electrons in Flatland", Scientific American, March 1996.

In quantum mechanics, identical particles are truly indistinguishable (Fig. 2). This means that the wavefunction for these particles must return to itself, up to an overall phase, when the particles are permuted:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \exp(\mathrm{i}\chi)\Psi(\mathbf{r}_2, \mathbf{r}_1, \cdots). \tag{5}$$

where $\cdots$ represents potentially many other identical particles.

We can illustrate this with a peek at an advanced topic mixing quantum field theory and relativity. Here is a scattering event of a photon off an electron, viewed in two reference frames; time is vertical, a spatial coordinate is horizontal. On the left we see two 'different' electrons, one which is created along with an anti-electron or positron e^+, and the other which later annihilates the positron. On the right we see the same event viewed in a different reference frame; here there is only one electron, which scatters two photons. (The electron is *virtual*, moving faster than light, between the collisions; this is allowed in intermediate states for quantum transitions.) The two electrons on the left are not only indistinguishable, they are the *same particle*! The antiparticle is also the electron, traveling backward in time.[7]

[7]This idea is due to Feynman's thesis advisor, John Archibald Wheeler. As Feynman quotes in his Nobel lecture, *I received a telephone call one day at the graduate college at Princeton from Professor Wheeler, in which he said, "Feynman, I know why all electrons have the same charge and the same mass." "Why?" "Because, they are all the same electron!" And, then he explained on the telephone, "suppose that the world lines which we were ordinarily considering before in time and space - instead of only going up in time were a tremendous knot, and then, when we cut through the knot, by the plane corresponding to a fixed time, we would see many, many world lines and that would represent many electrons, except for one thing. If in one section this is an ordinary electron world line, in the section in which it reversed itself and is coming back from the future we have the wrong sign to the proper time - to the proper four velocities - and that's equivalent to changing the sign of the charge, and, therefore, that part of a path would act like a positron."*

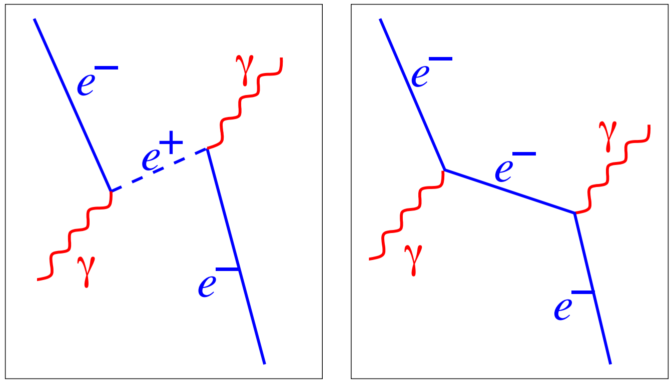

Fig. 2 Feynman diagram: identical particles.

In three dimensions, χ must be either zero or π, corresponding to bosons and fermions. In two dimensions, however, χ can be anything: *anyons* are possible! Let's see how this is possible.

In a two-dimensional system, consider changing from coordinates $\mathbf{r}_1, \mathbf{r}_2$ to the center-of-mass vector $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, the distance between the particles $r = |\mathbf{r}_2 - \mathbf{r}_1|$, and the angle ϕ of the vector between the particles with respect to the $\hat{x}$ axis. Now consider permuting the two particles counter-clockwise around one another, by increasing ϕ at fixed r. When $\phi = 180° \equiv \pi$, the particles have exchanged positions, leading to a boundary condition on the wavefunction

$$\Psi(\mathbf{R}, r, \phi, \cdots) = \exp(\mathrm{i}\chi)\Psi(\mathbf{R}, r, \phi + \pi, \cdots). \tag{6}$$

Permuting them counter-clockwise (backward along the same path) must then[8] give $\Psi(\mathbf{R}, r, \phi, \cdots) = \exp(-\mathrm{i}\chi)\Psi(\mathbf{R}, r, \phi - \pi, \cdots)$. This in general makes for a many-valued wavefunction (similar to Riemann sheets for complex analytic functions).

Why can't we get a general χ in three dimensions?

(a) *Show, in three dimensions, that* $\exp(\mathrm{i}\chi) = \pm 1$, *by arguing that a counter-clockwise rotation and a clockwise rotation must give the same phase.* (Hint: The phase change between ϕ and $\phi + \pi$ cannot change as we wiggle the path taken to swap the particles, unless the particles hit one another during the path. Try rotating the counter-clockwise path into the third dimension: can you smoothly change it to clockwise? What does that imply about $\exp(\mathrm{i}\chi)$?)

[8]The phase of the wave-function doesn't have to be the same for the swapped particles, but the gradient of the phase of the wavefunction is a physical quantity, so it must be minus for the counter-clockwise path what it was for the clockwise path.

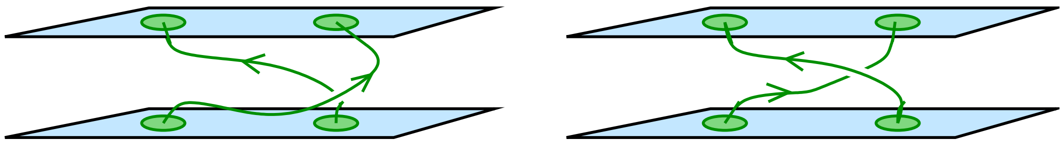

Fig. 3 Braiding of paths in two dimensions. In two dimensions, one can distinguish swapping clockwise from counter-clockwise. Particle statistics are determined by representations of the *Braid group*, rather than the permutation group.

Figure 3 illustrates how in two dimensions rotations by π and $-\pi$ are distinguishable; the trajectories form 'braids' that wrap around one another in different ways. You can't change from a counter-clockwise braid to a clockwise braid without the braids crossing (and hence the particles colliding).

An angular boundary condition multiplying by a phase should seem familiar: it's quite similar to that of the Bohm-Aharonov effect we studied in exercise 2.4. Indeed, we can implement fractional statistics by producing *composite particles*, by threading a magnetic flux tube of strength Φ through the center of each 2D boson, pointing out of the plane.

(b) *Remind yourself of the Bohm-Aharonov phase incurred by a particle of charge e encircling counter-clockwise a tube of magnetic flux Φ. If a composite particle of charge e and flux Φ encircles another identical composite particle, what will the net Bohm-Aharonov phase be?* (Hint: You can view the moving particle as being in a fixed magnetic field of all the other particles. The moving particle doesn't feel its own flux.)

(c) *Argue that the phase change* $\exp(\mathrm{i}\chi)$ *upon swapping two particles is exactly half that found when one particle encircles the other. How much flux is needed to turn a boson into an anyon with phase* $\exp(\mathrm{i}\chi)$*?* (Hint: The phase change can't depend upon the precise path, so long as it braids the same way. It's *homotopically invariant*, see chapter 9 of "Entropy, Order Parameters, and Complexity".)

Anyons are important in the quantum Hall effect. What is the quantum Hall effect? At low temperatures, a two dimensional electron gas in a perpendicular magnetic field exhibits a Hall conductance that is quantized, when the *filling fraction* ν (electrons per unit flux in units of Φ_0) passes near integer and rational values.

Approximate the quantum Hall system as a bunch of composite particles made up of electrons bound to flux tubes of strength Φ_0/ν. As a perturbation, we can imagine later relaxin the binding and allow the field to spread uniformly.[9]

[9] This is not nearly as crazy as modeling metals and semiconductors as non-interacting electrons, and adding the electron interactions later. We do that all the time – 'electons and holes' in solid-state physics, '1s, 2s, 2p' electrons in multi-electron atoms, all have obvious meanings only if we ignore the interactions. Both the composite particles and the non-interacting electron model are examples of how we use *adiabatic continuity* – you find a simple model you can solve, that can be related to the true model by turning on an interaction.

(d) **Composite bosons and the integer quantum Hall effect.** *At filling fraction $\nu = 1$ (the 'integer' quantum Hall state), what are the effective statistics of the composite particle? Does it make sense that the (ordinary) resistance in the quantum Hall state goes to zero?*

- The excitations in the *fractional* quantum Hall effect are anyons with fractional charge. (The $\nu = 1/3$ state has excitations of charge $e/3$, like quarks, and their wavefunctions gain a phase $\exp(i\pi/3)$ when excitations are swapped.)
- It is conjectured that, at some filling fractions, the quasiparticles in the fractional quantum Hall effect have *non-abelian* statistics, which could become useful for quantum computation.
- The composite particle picture is a centeral tool both conceptually and in calculations for this field.

4.6 **Superfluids: density matrices and ODLRO part 1.** (Condensed matter, Quantum) ⑤

(Optional: Extra credit.)

This exercise develops the quantum theory of the order parameters for superfluids and superconductors, following a classic presentation by Anderson (see, e.g. "Considerations on the flow of superfluid helium", Rev. Mod. Phys. **38**, 298). In this part of the exercise, we introduce the reduced density matrix and off-diagonal long-range order The exercise is challenging; it introduces creation and annihilation operators for fields, it involves technically challenging calculations, and the concepts it introduces are deep and subtle…

Density matrices. We saw in Exercise (2.5) that a Bose-condensed ideal gas can be described in terms of a complex number $\psi(\mathbf{r})$ representing the eigenstate which is macroscopically occupied. For superfluid helium, the atoms are in a strongly-interacting liquid state when it goes superfluid. We can define the order parameter $\psi(\mathbf{r})$ even for an interacting system using the *reduced density matrix.*

Suppose our system is in a mixture of many-body states Ψ_α with probabilities P_α. The full density matrix in the position representation, you will remember, is

$$\begin{aligned}\widehat{\rho}(\mathbf{r}'_1, \ldots, &\mathbf{r}'_N, \mathbf{r}_1, \ldots, \mathbf{r}_N) \\ &= \sum_\alpha P_\alpha \Psi^*(\mathbf{r}'_1, \ldots, \mathbf{r}'_N)\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N). \end{aligned} \tag{7}$$

(Properly speaking, these are the matrix elements of the density matrix in the position representation; rows are labeled by $\{\mathbf{r}'_i\}$, columns are labeled by $\{\mathbf{r}_j\}$.) The reduced density matrix $\widehat{\rho}(\mathbf{r}', \mathbf{r})$ (which I will call the density matrix hereafter) is given by setting

$\mathbf{r}'_j = \mathbf{r}_j$ for all but one of the particles and integrating over all possible positions, multiplying by N:

$$\begin{aligned}
&\widehat{\rho}_2(\mathbf{r}', \mathbf{r}) = \\
&\quad N \int \mathrm{d}^3 r_2 \cdots \mathrm{d}^3 r_N \\
&\qquad \times \widehat{\rho}(\mathbf{r}', \mathbf{r}_2 \ldots, \mathbf{r}_N, \mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N).
\end{aligned} \tag{8}$$

(For our purposes, the fact that it is called a matrix is not important; think of $\widehat{\rho}_2$ as a function of two variables.)

(a) *What does the reduced density matrix $\rho_2(\mathbf{r}', \mathbf{r})$ look like for a zero-temperature Bose condensate of non-interacting particles, condensed into a normalized single-particle state $\zeta(\mathbf{r})$? (That is, $\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{m=1}^{N} \zeta(\mathbf{r}_m)$.)*

An alternative, elegant formulation for this density matrix is to use second-quantized creation and annihilation operators instead of the many-body wavefunctions. These operators $a^\dagger(\mathbf{r})$ and $a(\mathbf{r})$ add and remove a boson at a specific place in space. They obey the commutation relations

$$\begin{aligned}
[a(\mathbf{r}), a^\dagger(\mathbf{r}')] &= \delta(\mathbf{r} - \mathbf{r}'), \\
[a(\mathbf{r}), a(\mathbf{r}')] &= [a^\dagger(\mathbf{r}), a^\dagger(\mathbf{r}')] = 0;
\end{aligned} \tag{9}$$

since the vacuum has no particles, we also know

$$\begin{aligned}
a(\mathbf{r})|0\rangle &= 0, \\
\langle 0|a^\dagger(\mathbf{r}) &= 0.
\end{aligned} \tag{10}$$

We define the ket wavefunction as

$$\begin{aligned}
|\Psi\rangle = (1/\sqrt{N!}) \int &\mathrm{d}^3 r_1 \cdots \mathrm{d}^3 r_N \\
&\times \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) a^\dagger(\mathbf{r}_1) \ldots a^\dagger(\mathbf{r}_N)|0\rangle.
\end{aligned} \tag{11}$$

(b) *Show that the ket is normalized if the symmetric Bose wavefunction Ψ is normalized.* (Hint: Use eqn 9 to pull the as to the right through the $a^\dagger$s in eqn 11; you should get a sum of $N!$ terms, each a product of N δ-functions, setting different permutations of $\mathbf{r}_1 \cdots \mathbf{r}_N$ equal to $\mathbf{r}'_1 \cdots \mathbf{r}'_N$.) *Show that $\langle \Psi|a^\dagger(\mathbf{r}')a(\mathbf{r})|\Psi\rangle$, the overlap of $a(\mathbf{r})|\Psi\rangle$ with $a(\mathbf{r}')|\Psi\rangle$ for the pure state $|\Psi\rangle$ gives the the reduced density matrix 8.*

Since this is true of all pure states, it is true of mixtures of pure states as well; hence the reduced density matrix is the same as the expectation value $\langle a^\dagger(\mathbf{r}')a(\mathbf{r})\rangle$.

In a non-degenerate Bose gas, in a system with Maxwell–Boltzmann statistics, or in a Fermi system, one can calculate $\widehat{\rho}_2(\mathbf{r}', \mathbf{r})$ and show that it rapidly goes to zero as

$|\mathbf{r}' - \mathbf{r}| \to \infty$. This makes sense; in a big system, $a(\mathbf{r})|\Psi(\mathbf{r})\rangle$ leaves a state with a missing particle localized around $\mathbf{r}$, which will have no overlap with $a(\mathbf{r}')|\Psi\rangle$ which has a missing particle at the distant place $\mathbf{r}'$.

ODLRO and the superfluid order parameter. This is no longer true in superfluids; just as in the condensed Bose gas of part (a), interacting, finite-temperature superfluids have a reduced density matrix with off-diagonal long-range order (ODLRO);

$$\widehat{\rho}_2(\mathbf{r}', \mathbf{r}) \to \psi^*(\mathbf{r}')\psi(\mathbf{r}) \quad \text{as } |\mathbf{r}' - \mathbf{r}| \to \infty. \tag{12}$$

It is called long-range order because there are correlations between distant points; it is called off-diagonal because the diagonal of this density matrix in position space is $\mathbf{r} = \mathbf{r}'$. The order parameter for the superfluid is $\psi(\mathbf{r})$, describing the long-range piece of this correlation.

(c) *What is $\psi(\mathbf{r})$ for the non-interacting Bose condensate of part (a), in terms of the condensate wavefunction $\zeta(\mathbf{r})$?*

This reduced density matrix is analogous in many ways to the density–density correlation function for gases $C(\mathbf{r}', \mathbf{r}) = \langle \rho(\mathbf{r}')\rho(\mathbf{r})\rangle$ and the correlation function for magnetization $\langle M(\mathbf{r}')M(\mathbf{r})\rangle$ (Chapter 10 of "Entropy, Order Parameters, and Complexity"). The fact that $\widehat{\boldsymbol{\rho}}_2$ is long range is analogous to the fact that $\langle M(\mathbf{r}')M(\mathbf{r})\rangle \sim \langle M\rangle^2$ as $\mathbf{r}' - \mathbf{r} \to \infty$; the long-range order in the direction of magnetization is the analog of the long-range phase relationship in superfluids.