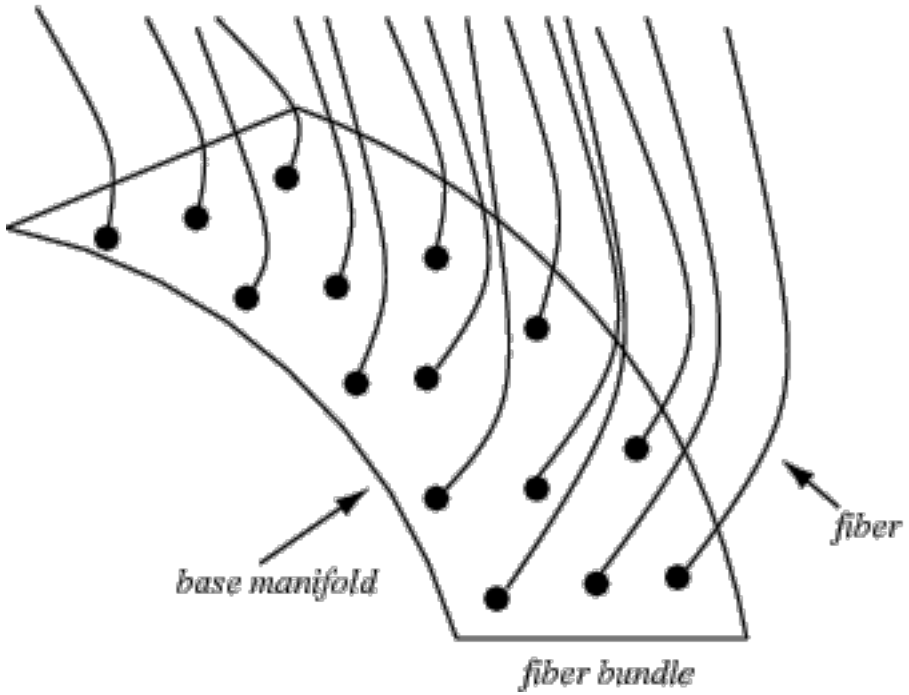
Parallel Transport in Physics

Gauge invariance, electromagnetism, falling cats, swimming bacteria, …



Fiber Bundles

Base space B

Real space

Fiber F at each b in B

Complex phase $e^{i\phi}$ of quantum wave function (point on circle)

Cross section f(b)

$e^{i\phi(x)}$ $(\psi(x) = |\psi(x)|\ e^{i\phi(x)})$

Covariant derivative D(f)

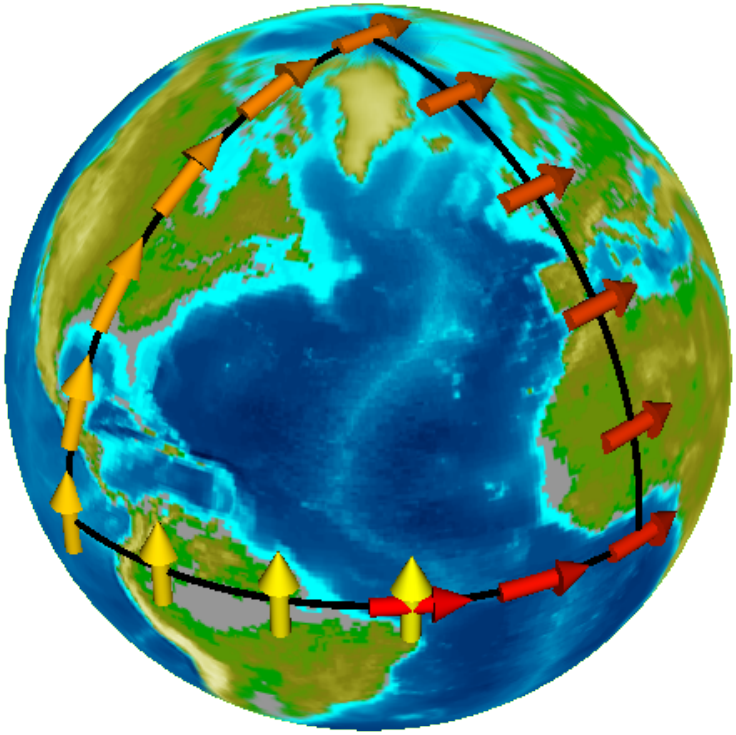
$$\left(\nabla - \frac{ie}{c}A\right)\psi$$

Curvature

B = curl A

Vectors on curved surfaces

Differential geometry



Parallel Transport

Make Dv=0 around closed loop

Base space

Surface of sphere (manifold)

Fiber F

Tangent vector *v*

Covariant derivative

$$(D_i v)^j = \partial_i v^j + \Gamma^j_{ik} v^k$$

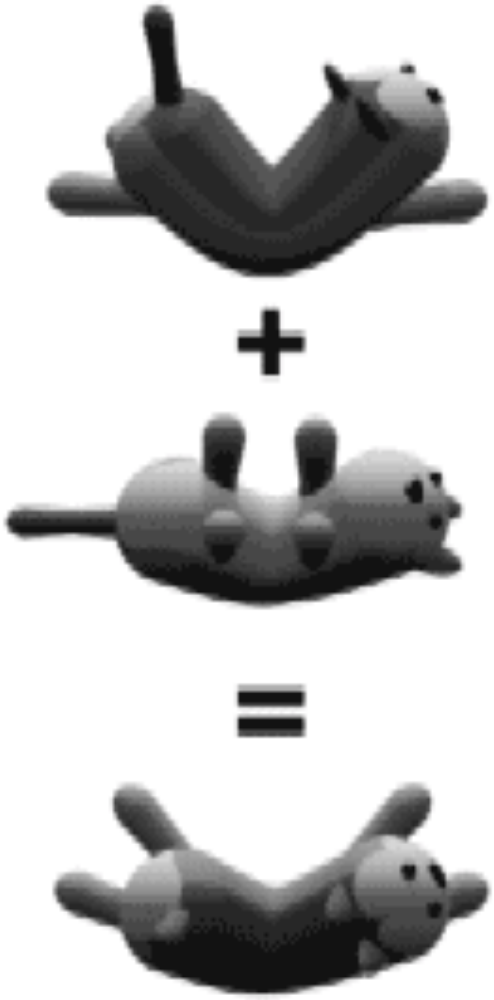
Curvature

Riemann tensor R^i_{jkl}

Basis for general relativity

Falling cats

Angular motion without angular momentum



Base space

Shape of cat

Fiber

Cat Orientation

Curvature

Closed loop in shape space

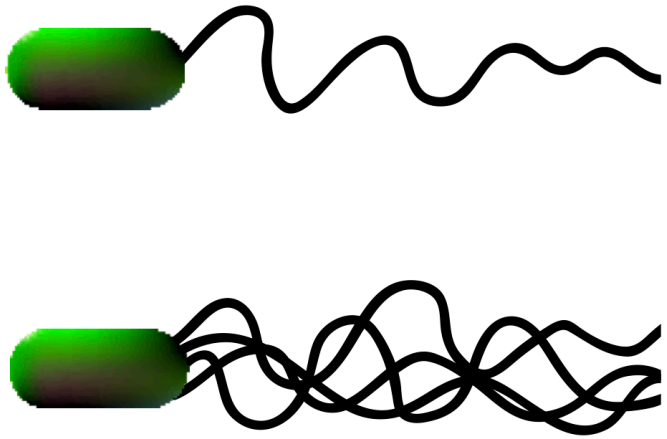
Cat reorients without angular momentum

Parallel Transport

No external torque; zero angular momentum

Swimming Bacteria

Life at low Reynold's number



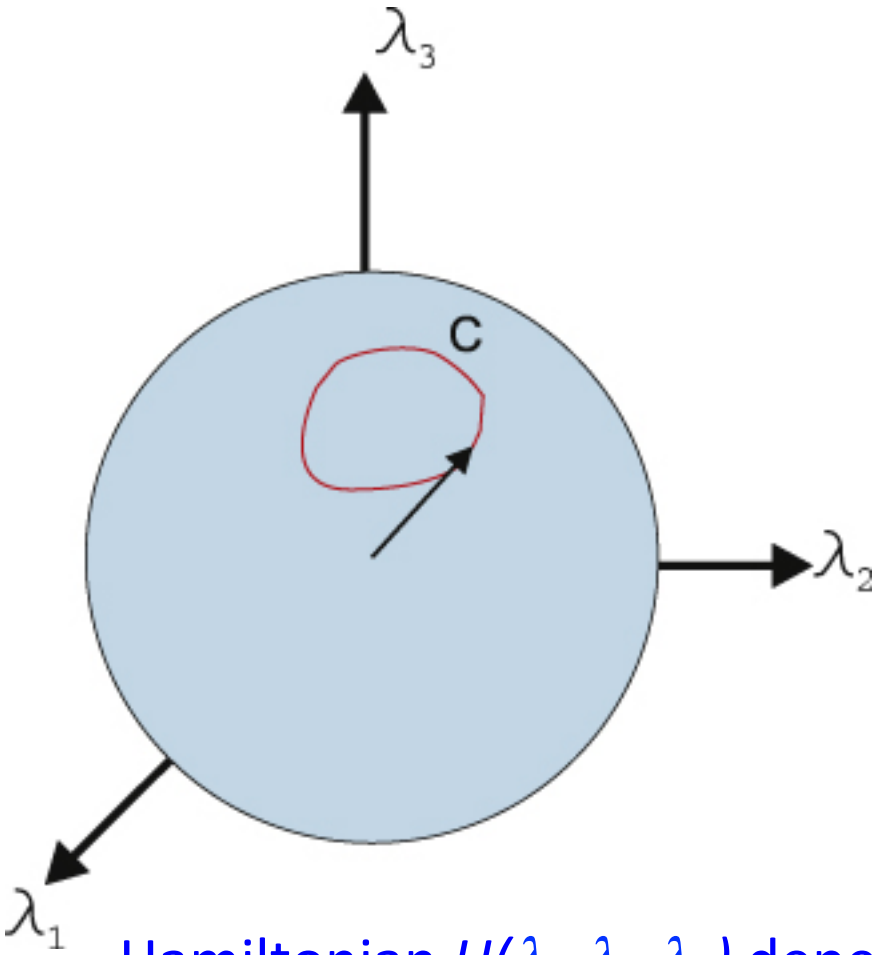
Flagellum:
'whip' that propels cells
Bacterial flagellum:
helical fragments that spin
Animal, plant, yeast (eukaryotes):
lash back and forth

Low Reynold's number, swimming very different. Like swimming through syrup or tar. "Kicking" or waving arms don't move you forward: only a closed loop in shape space allows propulsion.

Base Space: Shape. *Fiber:* Position in physical space.
Parallel Transport: swimming.

Berry's Phase

Quantum phase change for time-varying Hamiltonian



Base space

Space of all Hamiltonians

Fiber

Phase of ground state wavefunction

Covariant derivative

Slowly changing parameters leaves system in ground state up to phase (adiabatic theorem)

Parallel Transport

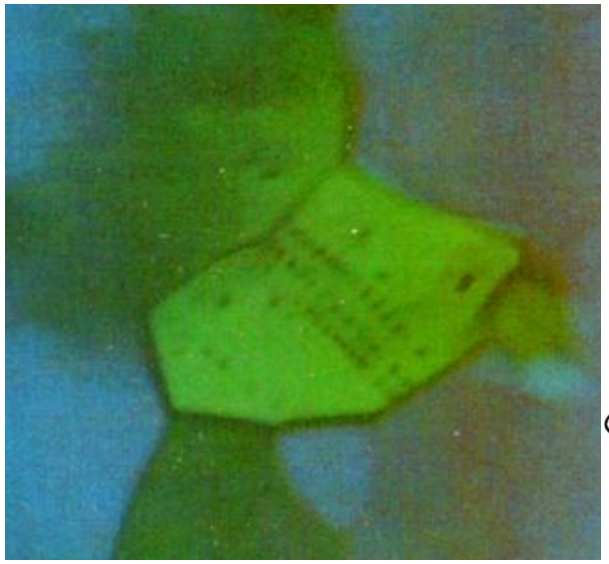
Phase change

Hamiltonian $H(\lambda_1,\lambda_2,\lambda_3)$ depends on parameters λ. Wave function evolves under slow change of $\lambda(t)$

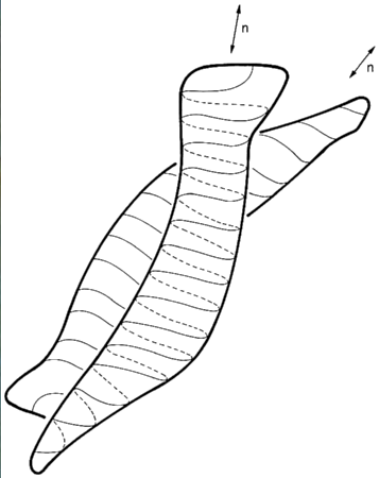
Phase of quantum wavefunction changes after closed loop

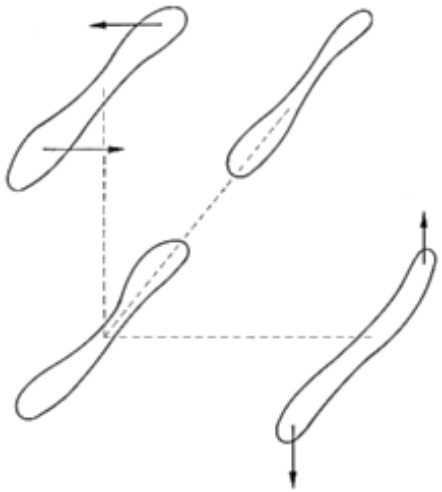
Liquid Crystals with Twist

The Blue phase and curved space



Blue from Bragg scattering off of cubic lattice of defect lines.





Double Twist

Base space

Physical space

Fiber

Molecular orientation of chiral molecules

Covariant derivative

Double twist: molecules like to sit at small angle

Curvature

Negative in flat space.

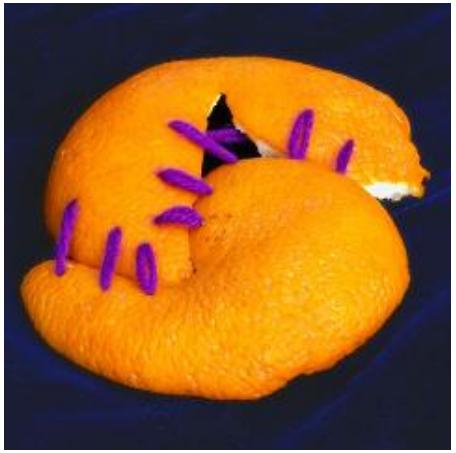
Frustration: can't fill space with double twist.

Zero on surface of sphere in four dimensions.

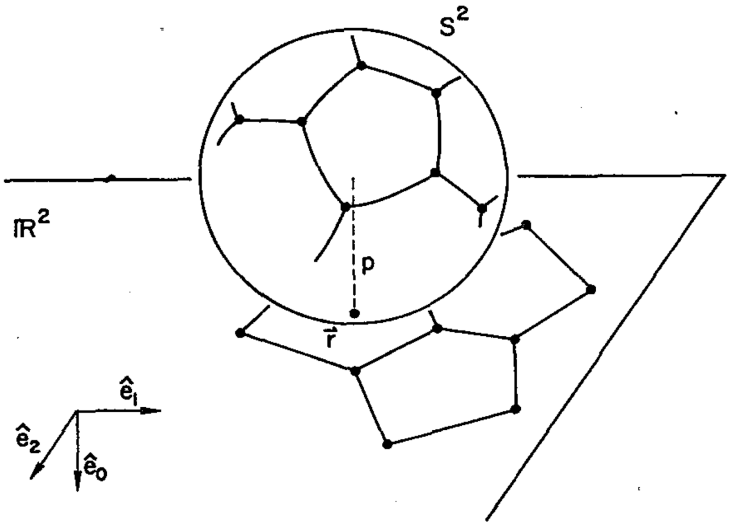
Frustrated materials

Metallic glasses and packing icosahedra

- Liquid metals like mercury prefer closed-packed tetrahedral order.
- Frustration: tetrahedra can't fit around edge
- Ideal template on sphere in 4D
- Parallel transport: rolling sphere
- Coupled translational, rotational order



Defect lines: cut sphere to flatten it.



Parallel Transport