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e→9

Electromagnetism and Quantum Mechanics

Sakurai 2.7 p. 135–136

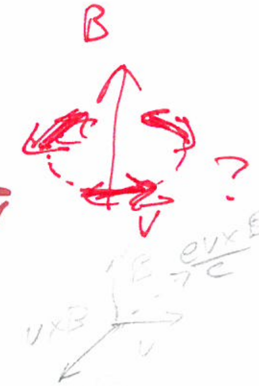
$E = -\nabla\phi$

Job: Motivate why E = ∇ d(phi)/dx and B = curl A produce Schrodinger's equation for the electron (e<0)

$$i\hbar \frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2 \psi + q\,\phi(x)\,\psi \qquad p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Plan: Show that this equation generates the Lorentz force

$$m\frac{d^2x}{dt^2} = \vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$$



Example: Constant B-field gives circular motion

$|a| = \omega_c^2 R = \frac{q}{mc}|v||B|$, $v = \omega_c R \Rightarrow \omega_c = \frac{qB}{mc}$ for all $\vec{v}$

ω_c: cyclotron frequency

Problem: x is an operator. Schrodinger says wave functions evolve, not operators!
Solution: Heisenberg representation.

Schrödinger: $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

$$\langle\varphi(t)|O|\psi(t)\rangle = \langle\varphi(0)|\underbrace{e^{iHt/\hbar}\,O\,e^{-iHt/\hbar}}_{\text{Heisenberg } O(t)}|\psi(0)\rangle$$

Heisenberg $O(t) = e^{iHt/\hbar}\,O(0)\,e^{-iHt/\hbar}$

$$\frac{dO}{dt} = \frac{iH}{\hbar}e^{iHt/\hbar}\,O\,e^{-iHt/\hbar} + e^{iHt/\hbar}\,O\,e^{-iHt/\hbar}\left(-\frac{iH}{\hbar}\right)$$

$$dO/dt = \frac{1}{i\hbar}[O,H]$$

$$dO/dt = \frac{1}{i\hbar}[O,H] \qquad H = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi(x)$$

Now we need some commutators. Remember

$$[x_i, p_j] = i\hbar\,\delta_{ij} \qquad [O, p^2] = \underbrace{[O,p]p + p[O,p]}_{OPP - \cancel{POP} + \cancel{POP} - PPO \text{ any } O,P}$$

$$[p, f(x)] = \underbrace{-i\hbar \vec{\nabla} f}_{\text{HW 2.1 or } p = \frac{\hbar}{i}\frac{\partial}{\partial x}}$$

$$dx/dt = \frac{1}{i\hbar}[x,H] = \frac{1}{i\hbar}\overbrace{\left[x, \frac{1}{2m}\left(p - \frac{e}{c}A(x)\right)^2\right]}^{[O,p^2]} + \frac{1}{i\hbar}\cancel{[x, e\phi(x)]}$$

$$= \frac{1}{2i\hbar m}\left\{\left(p - \frac{e}{c}A\right)[\underbrace{x, p}_{i\hbar} - \cancel{e/c\,A(x)}] + [\underbrace{x, p}_{i\hbar} - \cancel{e/c\,A}]\left(p - e/c\,A\right)\right\}$$

$$= \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

Define kinematical momentum $\Pi = p - \frac{q}{c}A$

$$[\Pi_i, \Pi_j] = \cancel{[p_i, p_j]} - \frac{q}{c}\left(\underbrace{[A_i, p_j]}_{i\hbar\partial_j A_i} + \underbrace{[p_i, A_j]}_{-i\hbar\partial_i A_j}\right) + \frac{q^2}{c^2}\cancel{[A_i, A_j]}$$

$$= \frac{i\hbar q}{c}\underbrace{(\partial_i A_j - \partial_j A_i)}_{\text{B field in 'third'}}$$

$$= \frac{i\hbar q}{c}\varepsilon_{ijk} B_k$$

$$\varepsilon_{ijk}\,\varepsilon_{klm}\nabla_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\nabla_l A_m \checkmark$$

$$\frac{d^2x_i}{dt^2} = \frac{1}{i\hbar}\left[\frac{dx_i}{dt}, H\right] = \frac{1}{i\hbar}\left[\frac{\Pi_i}{m}, \frac{\Pi^2}{2m} + e\phi\right]$$

$$= \frac{q}{mi\hbar}[\Pi_i, \phi] + \frac{1}{2m^2 i\hbar}\underbrace{[\Pi_i, \Pi_j^2]}_{[O,p^2]} \qquad \Pi_j^2 = \Pi_j\Pi_j = \sum_j \Pi_j^2 = \Pi^2 \text{ Einstein Summation}$$

$$= \frac{q}{mi\hbar}\left[p - \cancel{\frac{e}{c}A}, \phi\right] + \frac{1}{2m^2 i\hbar}\left(\underbrace{\Pi_j}_{m\,dx_j/dt}\underbrace{[\Pi_i, \Pi_j]}_{\times B_k\left(\frac{i\hbar e}{c}\right)} + [\Pi_i, \Pi_j]\Pi_j\right)$$

$$= -\frac{q}{m\cancel{i\hbar}}(\cancel{i\hbar}\underbrace{\nabla\phi}_{-E}) + \frac{1}{2m\cancel{i\hbar}}\left(\frac{\cancel{i\hbar} q}{c}\right)\left(\frac{dx}{dt}\times B - B\times\frac{dx}{dt}\right)$$

$$ma = m\frac{d^2x}{dt^2} = q\vec{E} + \frac{q}{2c}\left(\frac{dx}{dt}\times B - B\times\frac{dx}{dt}\right)$$

Classical Lorentz force:

$$m\frac{d^2x}{dt^2} = \vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$$

Quantum operator evolution:

$$ma = m\frac{d^2x}{dt^2} = q\vec{E} + \frac{q}{2c}\left(\frac{dx}{dt}\times B - B\times\frac{dx}{dt}\right)$$

Noncommuting version of $\vec{v}\times\vec{B}$

What does this mean? Electrons don't move with classical trajectories?

* Heisenberg picture operators evolve by Lorentz force
* Ehrenfest's theorem: expectation value of x in state |psi> evolves with Lorentz force. (Remember Heisenberg state |psi(t)> = |psi(0)> independent of time...)

$$\langle x\rangle \equiv \langle\psi|x(t)|\psi\rangle_{Heis} = \langle\psi(t)|x|\psi(t)\rangle_{Schr}$$

$$m\frac{d^2\langle x\rangle}{dt^2} = eE + \frac{e}{2c}\left(\frac{d\langle x\rangle}{dt}\times B - B\times\frac{d\langle x\rangle}{dt}\right)$$

$$m\langle a\rangle = eE + \frac{e}{c}\langle v\rangle\times B$$

✓

Saturat

why can I pull in $\langle\frac{dx}{dt}\times B\rangle \rightarrow \frac{d\langle x\rangle}{dt}\times B$??

$$\frac{e}{2cm}\left((p - \frac{e}{c}A)\times B - B\times(p - \frac{e}{c}A)\right)$$

$$\frac{e}{2mc}(p\times B - B\times p) - \frac{e^2}{mc^2}A\times B$$

✓ 4.207 Griffiths

Ehrenfest's theorem (ordinary):

$$m \frac{d^2}{dt^2}\langle x \rangle = \frac{d\langle p \rangle}{dt} = -\langle \nabla V(x) \rangle$$

<u>not</u> $-\nabla V(\langle x \rangle)$!

Motion of $\langle x \rangle$ depends on force smeared over wave packet

Analogously,

$$m \frac{d^2}{dt^2}\langle x \rangle = e\langle E \rangle + \frac{e}{2c}\left(\left\langle \frac{dx}{dt} \times B \right\rangle - \left\langle B \times \frac{dx}{dt} \right\rangle \right)$$

~~can't~~ not $eE(\langle x \rangle)$

<u>Not</u> $\frac{d\langle x \rangle}{dt} \times B$

or even $\left\langle \frac{dx}{dt} \right\rangle \times B$.