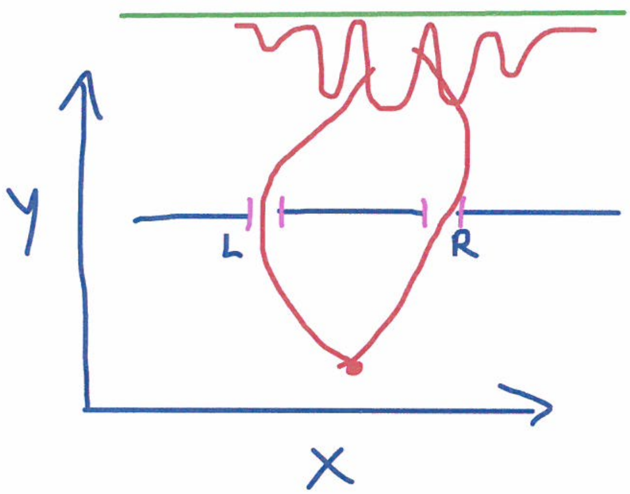
Path Integrals and the Classical Limit

Sakurai $\overset{2^{nd}}{\text{v}}$ p. 122 - 129 ; Feynman & Hibbs, Quantum Mechanics & Path Integrals



Amplitude on screen

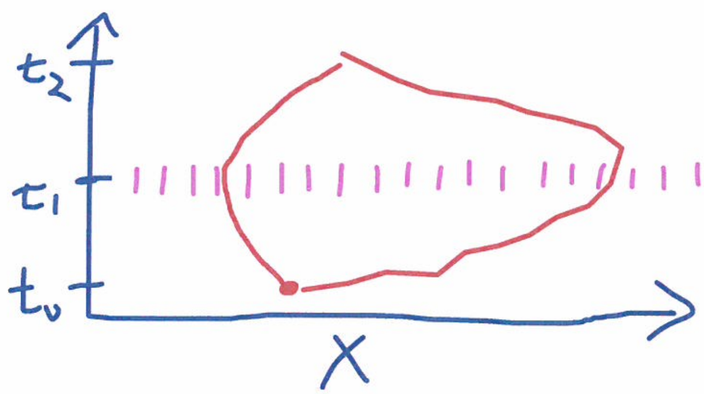
$\psi(x, y_{screen}) = \psi_R(x) + \psi_L(x)$

WF propagated from L, R slits

Quantum Superposition

Interference: Particle through <u>both</u> sides

No slits? Slits everywhere! Superimpose all possible states at intermediate time t1.



$$|\psi(t_2)\rangle = U(t_2 - t_1)\,|\psi(t_1)\rangle$$

$$\underbrace{\langle x_2|\psi(t_2)\rangle}_{\psi(x_2,t_2)} = \underbrace{\langle x_2|U(t_2-t_1)\overbrace{\int dx_1\,|x_1\rangle}^{\mathbb{1}}}_{\int dx_1 \underbrace{\langle x_2,t_2|x_1,t_1\rangle}_{\text{Propagator } K}} \underbrace{\langle x_1|\psi(t_1)\rangle}_{\psi(x_1,t_1)}$$

$$\psi(x_2,t_2) = \int dx_1\,\langle x_2,t_2|x_1,t_1\rangle\,\psi(x_1,t_1)$$

Heisenberg notation

$$\overbrace{\int dx_1}^{\text{Particle through "}x_1\text{ slit"}}\, K(x_2,t_2;x_1,t_1)\,\psi(x)$$

Which time? Every time!

$$\overset{\text{wrong}}{\cancel{\langle x_N, t_N | x_1, t_1 \rangle}} = \langle x_N | U(t_N - t_1) | x_1 \rangle$$

$$= \langle x_N | U(t_N - t_{N-1}) \mathbb{1}\, U(t_{N-1} - t_{N-2}) \mathbb{1} \dots \mathbb{1}\, U(t_2 - t_1) | x_1 \rangle$$

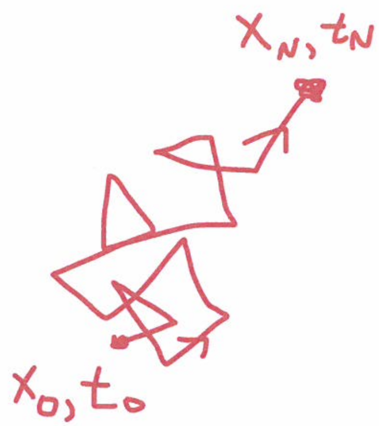
$$= \underbrace{\int dx_{N-1} \int dx_{N-2} \dots \int dx_2}_{\iint_{x_1, t_1}^{x_N, t_N} \mathcal{D}[x(t)]} \langle x_N | U(t_N - t_{N-1}) | x_{N-1} \rangle \langle x_{N-1} | \dots$$

$$\dots | x_2 \rangle \langle x_2 | U(t_2 - t_1) | x_1 \rangle$$

Path Integral

$$= \iint_{x_1\, t_1}^{x_N\, t_N} \mathcal{D}[x(t)] \cancel{\langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, \dots \langle x_2, t_2 | x_1, t_1 \rangle}$$

bad notation



Quantum: Particle takes all paths

Lagrangian Mechanics: Particle 'sniffs out' path of least action S — How does it do that? It tries all paths.

$$S = \int \tfrac{1}{2} m \dot{x}^2 - V(x)\, dt$$ } Same units [energy][time] as $\hbar$

[Do you really believe ~~...~~ Schrödinger? ... What derivation of QM would you believe?]

Feynman says: Time evolution as sum over paths

Phases evolve by $e^{iS[x(t)]/\hbar}$ — complex amplitude = magnitude × $e^{i\,\text{phase}}$

$$\underbrace{\langle x_N, t_N | x_0, t_0 \rangle}_{U(t_N - t_0)} = N \iint_{x(t_0)=x_0}^{x(t_N)=x_N} \mathcal{D}[x(t)]\, e^{iS[x(t)]/\hbar}$$

N = normalization. Ignore until Schrödinger.

Magnitude independent of path (how cool!)
phase: Must involve action. Must be dimensionless.

Why does this give the classical limit?

Big, heavy objects: S >> hbar. S varies rapidly as x(t) changes.

$$S(x(t) + \Delta(t)) - S(x(t)) \approx \int \Delta(t)\, \frac{\delta S}{\delta x}$$ $\Rightarrow$ Phase $\chi = iS/\hbar$ oscillates fast, cancels <u>unless</u> $\frac{\delta S}{\delta x} = 0$.

Lagrangian mechanics chooses path of least action, where $\delta S/\delta x = 0$.

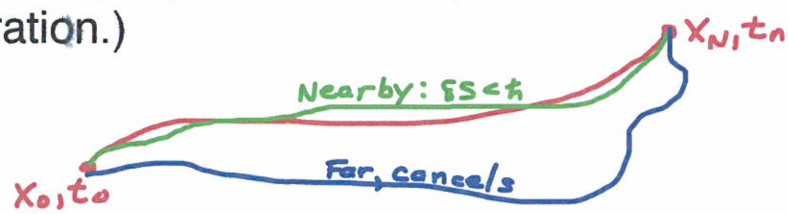
$$S(x+\delta) = \int dt\ \tfrac{1}{2} m (\dot{x} + \dot{\delta})^2 - V(x+\delta)$$ Quadratic

$$= \int \tfrac{1}{2} m \dot{x}^2 + m \dot{x} \dot{\delta} + \cancel{\tfrac{1}{2} m \dot{\delta}^2} - V(x) - \delta \underbrace{V'(x)}_{-F}\, dt$$

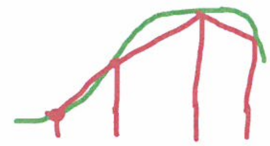
$$= S(x) + \int m \underbrace{\dot{x} \dot{\delta}}_{\text{By Parts}} + \delta \cdot F\, dt$$

$$S(x+\delta) - S(x) = \underbrace{\int (-m\ddot{x} + F)\, \delta(t)\, dt}_{\text{Zero for any } \delta(t) \text{ for minimum action}} \Rightarrow F = m\ddot{x}\ \checkmark$$

In QM, least action path & paths nearby contribute constructively; others oscillate rapidly and cancel. (See method of stationary phase in integration.)



Precise def'n of path integral:
Trapezoidal rule. Straight segments

$\int f(x) =$ 

$$\langle x+\Delta x | \overset{U(\Delta t)}{\cancel{t+\Delta t}} | x, \cancel{t} \rangle = \underbrace{N_{\Delta t}}_{N = \prod N_{\Delta t}^{n}} \exp\left[\frac{i}{\hbar}\left(\frac{1}{2} m \left(\frac{\Delta x}{\Delta t}\right)^2 - V(\bar{x})\right) dt\right]$$

Why does this give Schrodinger's equation?

$$\psi(y, t+\Delta t) = \int dx \, \langle y | \overset{U(\Delta t)|x\rangle}{\cancel{t+\Delta t | x, t}} \rangle \, \psi(x,t)$$

$$= \int dx \; N_{\Delta t} \exp\left[\frac{i}{\hbar}\left(\frac{1}{2} m \left(\frac{\overset{y-x}{\cancel{\Delta x}}}{\Delta t}\right)^2 - V(\bar{x})\right) dt\right] \psi(x,t)$$

$\xi = y - x$, small

$$= N_{\Delta t} \int d\xi \; e^{i/\hbar \left[\frac{1}{2} m \xi^2 / \Delta t - V(y - \cancel{\xi}\frac{\xi}{2}) \Delta t\right]} \; \underbrace{\psi(y-\xi, t)}_{\psi - \cancel{\xi \nabla \psi} - \frac{\xi^2}{2} \nabla^2 \psi}$$

odd

$$= \underbrace{e^{\frac{i}{\hbar}(-V \Delta t)}}_{1 + \frac{1}{i\hbar} V \Delta t} N_{\Delta t} \left\{ \psi \underbrace{\int e^{-iA\xi^2} d\xi}_{1/N_{\Delta t} = \sqrt{\pi / iA}} + \frac{1}{2} \nabla^2 \psi \underbrace{\int \xi^2 e^{-iA\xi^2} d\xi}_{\frac{1}{2}\sqrt{\pi/(iA)^3}} \right\}$$

(so $\psi(y, t+\Delta t) - \psi(y,t)$)

$$A = -\frac{m}{2\hbar \Delta t}; \quad N_{\Delta t} = \sqrt{\frac{iA}{\pi}} = \sqrt{-im/2\pi\hbar\Delta t};$$

$$\psi(y, t+\Delta t) = \psi(y, \Delta t) + \frac{\Delta t}{i\hbar} V \psi + \underbrace{N_{\Delta t}\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{\pi/(iA)^3}\right)}_{\frac{1}{4}\frac{1}{iA} = -\frac{\hbar \Delta t}{2mi} = \frac{\Delta t}{i\hbar}\left(-\frac{\hbar^2}{2m}\right)} \nabla^2 \psi$$

$$i\hbar \frac{\psi(y, t+\Delta t) - \psi(y)}{\Delta t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi \quad \checkmark$$

Path integrals:
* Usually harder to compute with (except WKB...)
* Important conceptual tool
* How the world works

Path integrals <u>in a field</u>

$$\sum e^{i\left(S/\hbar + \frac{1}{\hbar}\int \frac{q}{c}\vec{A}(x)\cdot d\ell\right)}$$

~~New~~ Adds phase $\frac{q}{\hbar c}\int A(x)\cdot d\ell$

to each path $x(t)$