Rotations and Spin

Evolve in time: $\underbrace{\mathcal{U}(t)}_{\text{Unitary}} = e^{-\frac{i}{\hbar}\underbrace{\mathcal{H}}_{\text{Hermitian}}t}$

Time independent $\Rightarrow$ Conserved Energy (Emmy Noether's Theorem)

Translate in Space: $\underbrace{\mathcal{D}(\vec{\Delta x})}_{\text{Unitary}} = e^{-\frac{i}{\hbar}\underbrace{\vec{p}}_{\text{Hermitian}}\cdot\vec{\Delta x}}$

Check: $p_x = -i\hbar\frac{\partial}{\partial x}$, $e^{-\frac{i}{\hbar}p_x\Delta x} = e^{-\Delta x\frac{\partial}{\partial x}}$ (Δx number, not operator)

Taylor expand $e^{\Delta x\frac{\partial}{\partial x}}\psi \approx \left(1 - \Delta x\frac{\partial}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2}{\partial x^2} + \dots\right)\psi$

$= \psi(x) - \Delta x\,\psi'(x) + \frac{\Delta x^2}{2}\psi''(x) + \dots$

$= \psi(x - \Delta x)$ ✓ translation to right

Translation invariance $\Rightarrow$ Conserved Momentum (Noether)

What about <u>rotations</u>?

Spin + Angular momentum conserved.

$L_z = i\hbar\frac{\partial}{\partial\phi}$

$S_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$ Hermitian.

Higgs

What is unitary operator which rotates spin ½ by φ about $\hat{z}$? Call it $\mathcal{D}_{\hat{z}}(\varphi)$.

Conjecture: $\mathcal{D}_{\hat{z}}(\varphi) = e^{-\frac{i}{\hbar} S_z \varphi}$

Q: If true, evaluate $\mathcal{D}_{\hat{z}}(\varphi)$, $\mathcal{D}_{\hat{z}}(\frac{\pi}{2})$, and $\mathcal{D}_{\hat{z}}(2\pi)$.

A: $\mathcal{D}_{\hat{z}}(\varphi) = e^{-\frac{i}{\hbar}\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}\varphi}$

$$= \cancel{e^{-i\varphi/2}}\ e^{\begin{pmatrix} -i\varphi/2 & 0 \\ 0 & i\varphi/2 \end{pmatrix}}$$ Already diagonal

$$= \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

$$\mathcal{D}_{\hat{z}}\left(\frac{\pi}{2}\right) = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = \begin{pmatrix} \sqrt{-i} & 0 \\ 0 & \sqrt{i} \end{pmatrix}$$

$$\mathcal{D}_{\hat{z}}(2\pi) = \begin{pmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$e^{-\frac{2i}{\hbar} S_z}$?

Oops. Are we off by a factor of two? Or do electrons & protons & neutrons have wavefunctions that change sign under 2π rotations?

Q: What should $|\uparrow_x\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ do under $\frac{\pi}{2}$ rotation about $\hat{z}$?
(No calculation.)

A: Should go to $|\uparrow_y\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$, or perhaps $|\downarrow_y\rangle$.

Q: Does it, given our conjecture?

A: $\begin{pmatrix} -\sqrt{i} & 0 \\ 0 & \sqrt{i} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{pmatrix} -\sqrt{i} \\ \sqrt{i} \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{i}} |\downarrow_y\rangle$

Up to a ~~consta~~ phase, this is OK!

Spin ½ particles <u>do</u> change ~~sign~~ wave function sign under 2π rotations.

[Feynman plate trick demo.]