Rotations: SU(2) and SO(3).

Spin wave function $\alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

rotate by φ about $\hat{z} \Rightarrow \mathcal{D}_z(\varphi)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$$\mathcal{D}_z(\varphi) = e^{-i/\hbar\, S_z \varphi}$$

What about $\mathcal{D}$ for general rotations?

3D rotation: axis $\hat{n}$ invariant

Define $\vec{S} = (S_x, S_y, S_z) = \hbar/2\,(\sigma_x, \sigma_y, \sigma_z) = \hbar/2\,\vec{\sigma}$

spin operators　　　Pauli matrices $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\mathcal{D}_{\hat{n}}(\varphi) = e^{-i/\hbar\, \vec{S}\cdot\hat{n}\,\varphi} =$$

$$= e^{-i\vec{\sigma}\cdot\hat{n}\,\varphi/2}$$

$\vec{\sigma}\cdot\hat{n} = \underbrace{\begin{pmatrix}n_z, & n_x - in_y\\ n_x+in_y, & -n_z\end{pmatrix}}_{\text{Hermitian}}$

$$= \mathbb{1} - i\frac{\vec{\sigma}\cdot\hat{n}}{2}\varphi - \frac{(\vec{\sigma}\cdot\hat{n}/2)^2}{2}\varphi^2 + i\frac{(\vec{\sigma}\cdot\hat{n}/2)^3}{3!}\varphi^3 + \ldots$$

Trick: $(\vec{\sigma}\cdot\hat{n})^2 = \mathbb{1}$, 　$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

$= \hat{n}_x^2\sigma_x^2 + \hat{n}_y^2\sigma_y^2 + \hat{n}_z^2\sigma_z^2 = 1$

$$= \mathbb{1}\left[1 - \frac{(\varphi/2)^2}{2} + \frac{(\varphi/2)^4}{4!} - \frac{(\varphi/2)^6}{6!}\ldots\right]$$

$$- i\vec{\sigma}\cdot\hat{n}\left[\varphi/2 - \frac{(\varphi/2)^3}{3!} + \frac{(\varphi/2)^5}{5!}\ldots\right]$$

$$= \mathbb{1}\cos\varphi/2 - i\vec{\sigma}\cdot\hat{n}\sin\varphi/2$$

$$\mathcal{D}_{\hat{n}}(\varphi) = \begin{pmatrix} \cos\varphi/2 - i n_z \sin\varphi/2 & -i\sin\varphi/2\,(n_x - i n_y) \\ -i\sin\varphi/2\,(n_x + i n_y) & \cos\varphi/2 + i n_z \sin\varphi/2 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \begin{aligned} a &= \cos\varphi/2 - i n_z \sin\varphi/2 \\ b &= (-n_y + i n_x)\sin\varphi/2 \end{aligned}$$

Cool Fact! If $a = x + iy$ and $b = z + i\omega$

then $\det \mathcal{D}_{\hat{n}}(\varphi)$ ~~then~~ $= |a|^2 + |b|^2 = x^2 + y^2 + z^2 + \omega^2 = 1$

~~easy~~ general to show

~~so $\mathcal{D}_{\hat{n}}(\varphi)$~~

$$\det \mathcal{D}_{\hat{n}}(\varphi) = aa^* - b(-b^*) = |a|^2 + |b|^2 \checkmark$$

$$|a|^2 + |b|^2 = (x+iy)(x-iy) + (z+i\omega)(z-i\omega) =$$
$$= x^2 + y^2 + z^2 + \omega^2 \checkmark$$

$$|a|^2 + |b|^2 = \cos^2\varphi/2 + n_z^2 \sin^2\varphi/2 + (n_y^2 + n_x^2)\sin^2\varphi/2$$
$$= \cos^2\varphi/2 + \underbrace{\hat{n}^2}_{1} \sin^2(\varphi/2) = 1$$

- $\mathcal{D}_{\hat{n}}(\varphi)$ is unitary [in $U(2)$, unitary 2×2 matrices] and special in that it has $\det \mathcal{D} = 1$

  $\Rightarrow SU(2)$

- The space of all $\mathcal{D}_{\hat{n}}(\varphi)$ is the unit sphere in four dimensions, S^3 ← 3D surface, sitting in higher (4D) space)

  S ← sphere

- What ~~are~~ about the ordinary rotation matrices SO(3)? $\begin{cases} S = \det 1 \text{ (no inversions)} \\ O = \text{orthogonal} \end{cases}$

~~Two matrices in SU(2)~~

Both SU(2) and SO(3) are <u>groups</u>.

- Set G
- (Maybe noncommutative) product $g_1 * g_2 \in G$
- Identity, inverse, associative

Two groups are isomorphic if invertible map $m: G_1 \to G_2$

$$m(g_1 * g_2) = m(g_1)\, m(g_2)$$

"coordinates" of rotation

$D^{1/2}_{\hat{n}}(\varphi)$ is <u>representation</u> of SU(2). $g_1 = (\hat{n}_1, \varphi_1)$ $g_2 = (\hat{n}_2, \varphi_2)$

~~Q! SU(2) and S~~

Representation = Map $G \to$ Matrices (Linear transforms on vectors), $R(g_1 * g_2) = R(g_1) * R(g_2)$

Q! Why do you know SU(2) and SO(3) are <u>not</u> isomorphic?

A! 2π rotation in SU(2) $= -\mathbb{I}$ $\quad D^{(\frac{1}{2})}(2\pi)$

maps to SO(3) $\underbrace{D^{(1)}(2\pi)}_{\text{Representation as } 3\times3 \text{ rotation matrices.}} = +\mathbb{I}$

<u>2-1 mapping</u>.