Superfluids, Superconductors, + the Anderson-Higgs Mechanism

* Quantum Mechanics invariant under overall change of phase:

$$\psi(\vec{r}) \to e^{i\frac{e}{\hbar c}\chi}\,\psi(\vec{r})$$

'Global' gauge symmetry

* Quantum Mechanics ~~a~~ invariant under local change of complex phase

$$\bar{\psi}(\vec{r}) = e^{i\underbrace{\frac{e}{\hbar c}\chi(\vec{r})}_{X(\vec{r})}}\,\psi(\vec{r})$$

'local' gauge symmetry

if $\frac{P^2}{2m} \to \frac{(\vec{P}-\frac{e}{c}\vec{\nabla}\chi)^2}{2m}$

Magnetic field $\vec{B}=\nabla\times\vec{A}$: $\frac{(P-\frac{e}{c}\vec{A})^2}{2m}$, $\vec{A}\to\vec{A}+\vec{\nabla}\chi$

Gauge invariance in E&M

~~Today~~: Next few days:

(1) What about many-particle wave functions?

(2) Noether: ~~conserved~~ $\to$ symmetry $\Leftrightarrow$ conservation law. What's conserved?

(3) Can we break this symmetry?

(4) What happens when charged bosons break gauge symmetry?

(5) How does this relate to Higgs?

(1) Many-body gauge invariance

$$\Psi(\vec{r}_1, \ldots, \vec{r}_N) = e^{i\chi(\vec{r}_1,\ldots,\vec{r}_N)}\,\Psi$$

How must $\chi(\vec{r}_1, \ldots, \vec{r}_N)$ behave?

Any two wavefunctions Ψ and Φ for the same particles must share the same phase change. Interference $\frac{1}{\sqrt{2}}(\Psi + \Phi)$ depends on relative phase at $\vec{r}_1, \ldots, \vec{r}_N$, must stay the same. $\chi_\Psi = \chi_\Phi$.

$$\text{Choose } \Phi(\underbrace{\vec{r}_1, \ldots, \vec{r}_M}_{US}, \underbrace{\vec{r}_{M+1} \ldots r_N}_{France}) = \overbrace{\underbrace{\phi_1(r_1) \ldots \phi_N(r_N)}_{\text{Product state}}}^{\text{Distinguishable}}$$

If $\vec{r}_1, \ldots, \vec{r}_M$ physically separated from $\vec{r}_{M+1}, \ldots, \vec{r}_N$, natural to assume $\chi_{TOT}(r_1 \ldots r_N)$

$$\chi_{TOT}(\vec{r}_1, \ldots, \vec{r}_N) = \chi_{US}(\vec{r}_1, \ldots \vec{r}_M) + \chi_{France}(r_{M+1}, \ldots, r_N),$$

One particle per country →

$$\Phi(r_1, \ldots, r_N) = e^{i\chi_1(r_1)}\phi_1(r_1)\, e^{i\chi_2(r_2)}\phi_2(r_2) \ldots$$

$$\rightarrow \chi(\vec{r}_1, \ldots, \vec{r}_N) = \sum \chi_j(r_j)$$

Take-home question: Is this 'natural' assumption true for anyons? For singular gauge transformations?

For indistinguishable particles

$$\Psi(\ldots\vec{r}_{e_i}, \ldots, \vec{r}_{e_j}, \ldots, \vec{r}_{p_1}, \ldots, \vec{r}_{n_1}, \ldots)$$

$$= \overset{\text{boson}}{\pm}\ \Psi(\ldots\vec{r}_{e_j}, \ldots, \vec{r}_{e_i}, \ldots, \vec{r}_{p_1}, \ldots, \vec{r}_{n_1} \ldots)$$

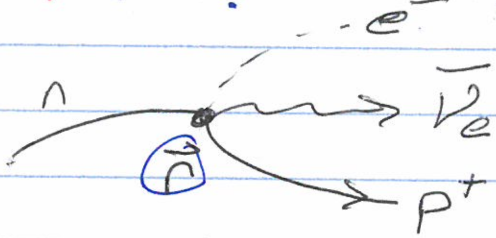
$$\Rightarrow \chi_{e_i}(\vec{r}) = \chi_{e_j}(\vec{r})$$

$\Rightarrow \chi_\alpha(\vec{r})$ depends only on particle type α.

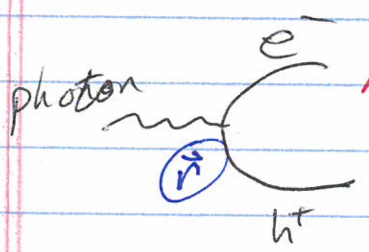
Remember: for E&M, $\chi_\alpha(r) = \frac{q_\alpha}{\hbar c}\chi(r)$ does depend on charge q_α of particle.

(2) What's the conservation law for gauge symmetry?

Problem: Suppose the number of particles changes?



Neutron decay



quasi electron - quasi hole pair creation in semiconductor

Particle creation/annihilation at one position $\vec{r}$ must not change phase

Q: Are these obeyed by $X_{E\&M} = \frac{q}{\hbar c}\chi(\vec{r})$? What's conserved?

$$X_n(\vec{r}) = X_{e^-}(\vec{r}) + X_{\bar{\nu}_e}(\vec{r}) + X_{p^+}(\vec{r})$$

$$X_{photon}(\vec{r}) = X_{e^-}(\vec{r}) + X_{h^+}(\vec{r})$$

A: Charge conservation $\Leftrightarrow$ E&M gauge invariance

$$\boxed{X_\alpha = \frac{q_\alpha}{\hbar c}\chi_{E\&M}(\vec{r})}$$

Helium atoms conserved (usually)

$\cancel{X_\alpha(\vec{r}) =}$

Number conservation $\Leftrightarrow$ Another gauge invariance

Strong interaction invariant under SU(3) rotation ('non-abelian gauge symmetry')

(like E&M invariant under $e^{i\chi}$ ⇔ U(1) symmetry)

SU(3) gauge invariance ⇔ Conservation of 'color' charge

photons (E&M gauge bosons) ⇔ gluons (massless bosons)

Weak interaction invariant under both U(1) and SU(2)

~~but no extra conserved quantities!~~

SU(2) spontaneously broken symmetry ⇔ Higgs mechanism.

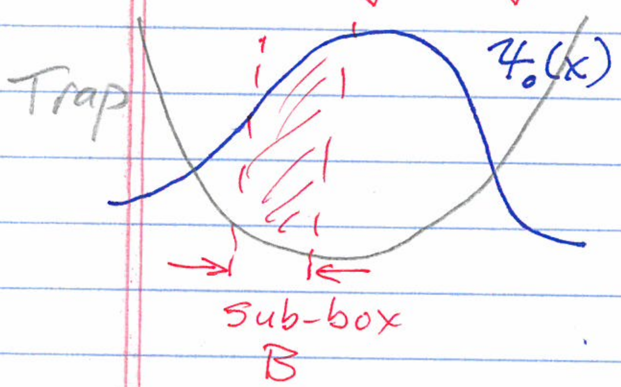
Angular momentum not conserved in magnet: $\vec{H}$ breaks rotation symmetry.

Can we find an analogy?

Superfluids & Superconductors also break gauge symmetry.

(3) Can we <u>break</u> gauge symmetry?

Bose gas ground state (cold atom trap), nearly non-interacting.



$\Psi(\vec{r}_1, \ldots, \vec{r}_N) = \psi_0(r_1) \ldots \psi_0(r_N)$

Total # of atoms $= N$

<u>fixed</u> → gauge invariant.

Why is this a superfluid? → over

However, each particle is delocalized over entire trap.

Consider a sub-box B. Let

$$|v|^2 = \int_B |\psi_0(r)|^2 dr, \quad |u|^2 = 1 - |v|^2$$

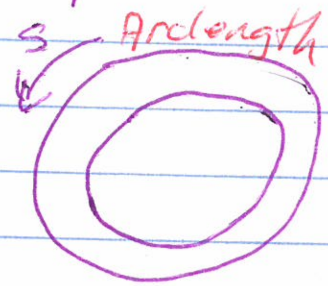
Q: What is the probability that $M < N$ particles are in B?

A: $\binom{N}{M} |v^2|^M |u^2|^{N-M} =$

$$= \frac{N!}{M!(N-M)!} |v^2|^M |u^2|^{N-M}$$

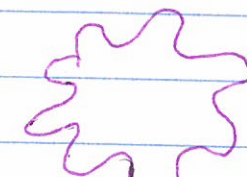
Number of particles in sub-box B is indeterminate.

Why is Ψ a superfluid?



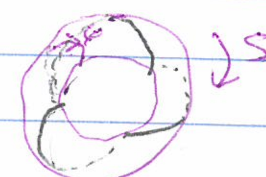
Supercurrent around a ring (superfluid or superconductor)

$$\psi_0(s) = |\psi| e^{iks} e^{i\omega_k t}$$



$$\Psi(s_1, s_2, \ldots) = \psi_0(s_1)\,\psi_0(s_2)\ldots$$

$$= |\psi|^N e^{iks_1} e^{iks_2}$$



$\varphi = ks$ winds three times

Bosons are gregarious

→ $\psi_i(s_i)$ all must change together

Superfluidity → It costs $|\psi|$ energy to change amplitude

Topology → Winding # can't change

$$J(s) = \frac{\hbar}{2mi} \int \overbrace{\sum_n}^{\text{for all } n} \underbrace{\left(\Psi^* \nabla_n \Psi - \Psi \nabla_n \Psi^*\right)}_{\text{current from particle } n} \underbrace{\delta(s - s_n)}_{\text{at } s}\, d^N s$$

$$= \frac{\hbar}{2mi} \sum_n \overbrace{\prod_{j \neq n} \int |\psi(s_j)|^2 ds_j}^{1} \int ds_n\, \psi_0^* \nabla \psi_0 - \psi_0 \nabla \psi_0^*$$

$$= N \frac{\hbar}{2mi} \int ds\, |\psi_0|^2\, e^{-iks} (ik) e^{iks} - e^{iks} (-ik) e^{-iks}$$

$$= N \frac{\hbar k}{m}$$

How can I describe Ψ^B, the 'wavefunction in B'?

Let $\psi_B(x) = \psi_0(x)/V$, the normalized ground state in the box.

$$\Psi^B = \prod_i (u + v\,\psi_B(r_i))$$

$$= u^N + u^{N-1} v \underbrace{\left[\psi_B(r_1) + \psi_B(r_2) + \ldots\right]}_{N = \binom{N}{1},\ \text{symmetrized}}$$

$$+ u^{N-2} v^2 \underbrace{\left[\psi_B(r_1)\psi_B(r_2) + \psi_B(r_1)\psi_B(r_3) + \ldots\right]}_{\text{Two-particle, symmetrized}}$$

$$= \Psi_0^B + \Psi_1^B + \Psi_2^B + \ldots + \Psi_N^B$$

= coherent superposition of states w/ different numbers of particles!

Let $\psi_B(r_1) = e^{i\phi(r_1)} |\psi_B(r_1)|$.

$\phi(r_1)$ is the relative phase between different particle # sectors of the wave function.

Just as a magnet picks out a direction $\vec{M}(\vec{r})$ (breaks rotational symmetry), so a superfluid picks out a phase $\phi(\vec{r})$ (breaks gauge symmetry).

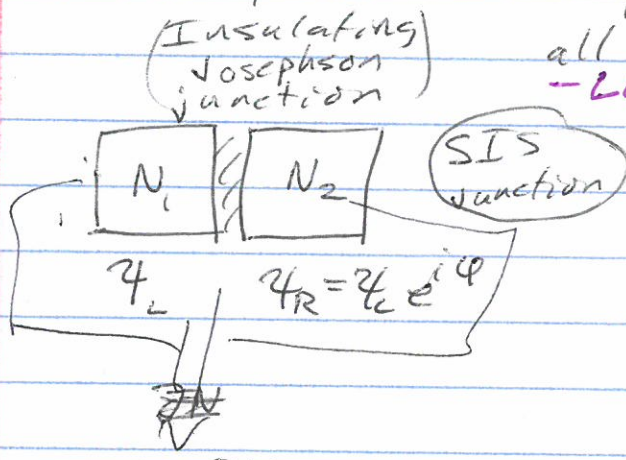
Higgs 7.5

Fix This: Josephson currents?

Number ~~and~~ Phase ~~as~~ uncertainty
(P.W. Anderson, Reviews of Modern Physics 38, 298 1966.)

Fixed number = Only one sector, no info on phase

Fixed phase = Superposition of all number states



-LEAP Here-

$$[N, \phi] = 1$$

like $[x, p] = i\hbar$

$$\left\{ \begin{array}{l} N \Leftrightarrow -i\,\partial/\partial\varphi \\ i\frac{\partial}{\partial N} \Leftrightarrow \phi \end{array} \right\}$$

$$i\hbar \frac{\partial N_1}{\partial t} = [\mathcal{H}, N] = i\partial\mathcal{H}/\partial\varphi$$ Josephson relation

$$\frac{\partial \phi}{\partial t} = V \Rightarrow N \sim \text{oscillates}$$

AC Josephson

What happens when charged bosons ~~condense~~? (Cooper pair of electrons $\approx$ boson)

break gauge symmetry? Superconductivity.

Charged $\rightarrow$ couple to gauge field $\vec{A}$.

Condensed $\rightarrow$ break gauge invariance.

Both?

Assume $\Psi(\vec{r}_1, \dots, \vec{r}_N) = \psi_0(\vec{r}_1)\psi_0(\vec{r}_2)\dots\psi_0(\vec{r}_N)$ with the ~~condensed state~~ $\psi_0(r)$ real $(\psi_0^* = \psi_0)$.

Q1: If the bosons are non-interacting, is the energy for Ψ = N times the single-particle energy E_0 for ψ_0?

$$\mathcal{H}_{\text{non-int}} = \sum -\frac{\hbar^2}{2m}\left(\nabla_i - \frac{iq}{\hbar c}A_i\right)^2 + \underset{\text{trap}}{V(r_i)} = \sum \mathcal{H}_i$$

A1: Yes. $\langle \Psi | \mathcal{H}_{NI} | \Psi \rangle = \sum_i \langle \Psi | \mathcal{H}_i | \Psi \rangle$

$$= \sum_{i=1}^{N} \int \cancel{\psi^*(r_1)} \dots \mathcal{H}_i \; \cancel{\psi(r_1)} \dots \psi(r_N) \; \cancel{dr_1} \dots dr_N$$

$$= \sum_{i=1}^{N} \int \psi^*(r_i)\, \mathcal{H}_i\, \psi(r_i)\, dr_i = NE_0.$$

Q2: Does the ~~single-particle~~ energy $E = NE_0$ depend on $\vec{A}$? If so, calculate it. (Now you'll use $\psi^* = \psi$).

A2: $\langle E \rangle = N \langle E_0 \rangle$

independent of A

$$= N \left\{ -\frac{\hbar^2}{2m} \int \psi_0^* \underbrace{\left(\nabla - \frac{iqA}{\hbar c} \right)^2}\psi_0 \, dr + \cancel{\int \psi^* V \psi \, dr} \right\}$$

why do these cancel?

$$\int dr \; \cancel{\psi_0^* \nabla^2 \psi_0} \; \cancel{- \frac{iqA}{\hbar c} \psi_0^* \nabla \psi_0} \; \cancel{- \frac{iq}{\hbar c} \psi_0^* \nabla (A \psi_0)}$$

independent of A

$$- \frac{q^2 A^2}{\hbar^2 c^2} |\psi_0|^2$$

imaginary energy? Must be zero.

$$\int dr \; \psi_0^* \nabla (A \psi_0) = - \underbrace{\int dr \left(\nabla \psi_0^* \right) A \psi_0}_{\psi \text{ is } \underline{\text{real}}}$$

integrate by parts

$$= - \int dr \; A \psi_0^* \nabla \psi_0$$

$$\langle E \rangle = N \left(-\frac{\hbar^2}{2m} \right) \int \left(-\frac{q^2 A^2}{\hbar^2 c^2} |\psi_0|^2 \right) dr$$

$$= N \int |\psi_0|^2 \frac{q^2}{2mc^2} \vec{A}^2 \, dr$$

~~Q3: What does this do to the photon?~~

~~[Simpler case: wave on string, displacement u w/spring~~

~~$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + d^2 u \text{ : solutions?}$$~~

~~]~~

What does this do to the photon?

Simpler case: wave on a string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Add energy $= \int \frac{1}{2} K u^2 \, dr$ (spring $F=-kx$)

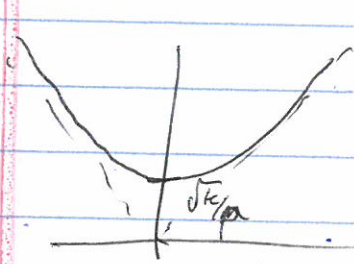
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \frac{K}{\rho} u$$

Q: What is $\omega(k)$ for this string-on-a-spring?

plug in $u_0 e^{i(kx-\omega t)}$

A: $-\omega^2 u_0 = -k^2 c^2 u_0 - \frac{K}{\rho} u_0$

$$\omega = \sqrt{ck + \frac{K}{\rho}}$$



High-energy physics: photon gains a <u>mass</u>

Q: How will a DC magnetic field behave in a ~~super~~ condensate of charged bosons?
(solve for $k(\omega)$ at $\omega=0$; use our string-on-a-spring)

A: $k = \sqrt{\frac{-Kc^2}{\rho}}$ imaginary $\rightarrow$ decays or grows as $e^{-\lambda x}$

with $\lambda = \sqrt{\frac{K}{\rho}}\, c$. <u>Meissner effect</u>: Superconductors expel ~~repel~~ magnetic fields.

(5) How does this relate to Higgs?

Electromagnetism	Electroweak
$e^{i\chi}\psi$	$u\vec{\psi}$
U(1) symmetry	SU(2) [×U(1)] symmetry (Nonabelian gauge Field
Condensate chooses phase $e^{i\phi}$ $\psi = \lvert\psi\rvert e^{i\varphi}$, $\Psi = \pi\psi$	Condensate chooses element of SU(2)

Spontaneously Broken Gauge Symmetry

Electromagnetism	Electroweak
Light	4 kinds of light
Meissner/Anderson	Higgs/.../
Photons Develop gap = 'mass'	3/4 kinds become massive 'Intermediate vector bosons' Z, $W^{\pm}$
(Nothing obvious)	Higgs Boson