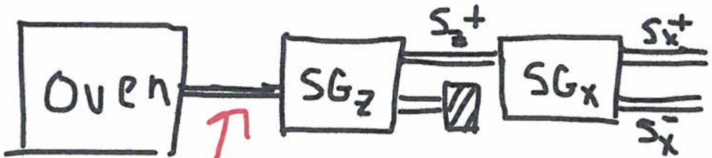
Density Matrices

Describe 'mixed' spin state of Stern Gerlach oven beam



'half' up & 'half' down

- Unpolarized
- Entangled after B observes
- Stat Mech

What is an entangled beam once it decoheres?

Stop using wavefunctions. Use 'density matrices'.
(Matrix has room for mixtures of several quantum states)

Wavefunction $\psi(x) = |\psi\rangle$

becomes 'pure state' density matrix $|\psi\rangle\langle\psi| = \rho$

Projection operator, taking component of $|\varphi\rangle$ along $|\psi\rangle$

$$\rho|\varphi\rangle = |\psi\rangle\underbrace{\langle\psi|\varphi\rangle}_{\text{Amplitude along } |\psi\rangle}$$

(phase information lost, ...)

ρ_{diag} vs. $\rho_{incoherent}$?

Define Coherence (Off-diagonal terms)

Contains all physical information?

$$\psi(x) = \langle x|\psi\rangle, \text{ so } \langle x|\rho|x'\rangle = \langle x|\psi\rangle\langle\psi|x'\rangle$$
$$= \psi(x)\psi^*(x')$$

Fix x' s.t. $\psi^*(x') = A \neq 0$.

$\langle x|\rho|x'\rangle = A\psi(x)$ (unknown A).

Normalize, $\int dx\, |\langle x|\rho|x'\rangle|^2 = \int dx\, |A|^2 |\psi(x)|^2 = |A|^2$

$$\frac{\langle x|\rho|x'\rangle}{\sqrt{\int |\langle x|\rho|x'\rangle|^2}} = e^{i\varphi}\psi(x), \text{ unknown phase } \varphi.$$

Overall phase of WF is not measurable (only phase differences):
Pure state density matrix has all info about state

2nd level of probability: ensemble has probabilities p_n of being in many orthogonal quantum states psi_n:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

Is QM awkward with density matrices? Slightly...

Time evolution: $$\frac{\partial\rho}{\partial t} = \sum_n p_n \Big(\underbrace{\frac{\partial|\psi_n\rangle}{\partial t}}_{\frac{1}{i\hbar}H|\psi_n\rangle} \langle\psi_n| + |\psi_n\rangle \underbrace{\frac{\partial\langle\psi_n|}{\partial t}}_{-\frac{1}{i\hbar}\langle\psi_n|H} \Big)$$

$$= \frac{1}{i\hbar}(H\rho - \rho H) = \frac{1}{i\hbar}[H,\rho] \qquad \frac{\partial\psi_n^*}{\partial t} = \left(\frac{\partial\psi_n}{\partial t}\right)^*$$

Note: opposite of Heisenberg picture

$$\frac{\partial O}{\partial t} = \frac{1}{i\hbar}[O,H]$$

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[H,\rho] = -\frac{1}{i\hbar}[\rho,H]$$

Makes sense!

Heisenberg $-\ \frac{\partial\psi}{\partial t} = 0$

$\Rightarrow \frac{\partial\rho}{\partial t} = 0$

Operator Expectation: $\langle\psi|O|\psi\rangle = Tr(O\rho)$

$Tr(M) = \sum M_{ii}$, independent of basis

Why? Try $|x\rangle$ basis

$$Tr(O\rho) = \sum_n p_n Tr(O|\psi\rangle\langle\psi|)$$

$$= \sum_n p_n \int dx \underbrace{\langle x|O|\psi\rangle}_{(O\psi)(x)} \underbrace{\langle\psi|x\rangle}_{\psi^*(x)}$$

$$= \sum_n p_n \int dx\, \psi^*(x)\, O\,\psi(x)\, dx$$

$$= \sum_n p_n \langle\psi_n|O|\psi_n\rangle$$

Observation by O: Weird! Not Schrodinger eqn.

Pure state $|\psi\rangle \to$ Mixture $|o_\alpha\rangle$ Just 'before' measurement

prob $p_\alpha = |\langle o_\alpha|\psi\rangle|^2$

Pure State:

$\rho = |\psi\rangle\langle\psi|$ in basis $|o_\alpha\rangle$

$$\rho = \mathbb{1}\rho\mathbb{1} = \left(\sum_\alpha |o_\alpha\rangle\langle o_\alpha|\right)\rho\left(\sum_\beta |o_\beta\rangle\langle o_\beta|\right)$$

$$= \sum_{\alpha,\beta} |o_\alpha\rangle\langle o_\alpha|\psi\rangle\langle\psi|o_\beta\rangle\langle o_\beta|$$

$$\rho_{\alpha\beta} = \langle o_\alpha|\psi\rangle\langle o_\beta|\psi\rangle^*$$

After Observation, Mixture:

$$\rho' = \sum p_\alpha |o_\alpha\rangle\langle o_\alpha|$$

$$= \sum |\langle o_\alpha|\psi\rangle|^2 |o_\alpha\rangle\langle o_\alpha|$$

$$\rho'_{\alpha\alpha} = \langle o_\alpha|\psi\rangle\langle o_\alpha|\psi\rangle^* = \rho_{\alpha\alpha}$$

$$\rho'_{\alpha\beta} = 0 \quad \alpha \neq \beta \text{ !}$$

Observation by 'classical' instrument destroys off-diagonal 'coherence'!

Back to S-G beam! Density matrix for singlet pure state:

$$|s\rangle\langle s| = \rho =$$

$$= \tfrac{1}{2}\left(|\uparrow\rangle_L |\downarrow\rangle_R - |\downarrow\rangle_L |\uparrow\rangle_R\right)\left({}_L\langle\uparrow|\,{}_R\langle\downarrow| - {}_L\langle\downarrow|\,{}_R\langle\uparrow|\right)$$

$$= \tfrac{1}{2}\,|\uparrow\rangle_L|\downarrow\rangle_R\,{}_L\langle\uparrow|\,{}_R\langle\downarrow| - \tfrac{1}{2}\,|\uparrow\rangle_L|\downarrow\rangle_R\,{}_L\langle\downarrow|\,{}_R\langle\uparrow|$$
$$- \tfrac{1}{2}\,|\downarrow\rangle_L|\uparrow\rangle_R\,{}_L\langle\uparrow|\,{}_R\langle\downarrow|$$
$$+ \tfrac{1}{2}\,|\downarrow\rangle_L|\uparrow\rangle_R\,{}_L\langle\downarrow|\,{}_R\langle\uparrow|$$

In the basis $\begin{pmatrix}\uparrow\uparrow\\ \uparrow\downarrow\\ \downarrow\uparrow\\ \downarrow\downarrow\end{pmatrix}$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Coherence between L&R in off-diagonal -1/2

As atoms separate, exposed to environment, coherence lost
Or, just 'forget' about L atoms!

Implement as Partial Trace...

Agree never to measure atoms left behind in oven (L-atoms).
All observables expandable in R basis:

$$O = \sum_{\substack{r_\alpha \in \uparrow_R, \downarrow_R \\ r_\beta \in}} O_{\alpha\beta}\, |r_\alpha\rangle_R\, {}_R\langle r_\alpha|$$

Expectation value

$$Tr(O\rho) = \sum_{r_\alpha}\sum_{l_\mu} {}_R\langle r_\alpha|\,{}_L\langle l_\mu|\,O\rho\,|l_\mu\rangle_L\,|r_\alpha\rangle_R$$

$$= \sum_{r_\alpha} \langle r_\alpha|\,O\,\underbrace{\Big(\sum_{l_\mu} {}_L\langle l_\mu|\rho|l_\nu\rangle_L\Big)}_{\tilde{\rho}\,=\,\text{Partial Trace over Forgotten Info}}\,|r_\alpha\rangle$$

Partial Trace

$$\tilde{\rho} = \sum_{\mu} \langle \ell_\mu | \rho | \ell_\nu \rangle$$

Contains all information necessary to predict R-atom observables.

Beam from S-G oven?

$$\rho = \tfrac{1}{2}\, |\uparrow\rangle_L |\downarrow\rangle_R \, {}_L\langle\uparrow| \, {}_R\langle\downarrow| - \tfrac{1}{2} |\uparrow\rangle_L |\downarrow\rangle_R \, {}_L\langle\downarrow| \, {}_R\langle\uparrow| \,)$$
$$- \tfrac{1}{2} |\downarrow\rangle_L |\uparrow\rangle_R \, {}_L\langle\uparrow| \, {}_R\langle\downarrow| + \tfrac{1}{2} |\downarrow\rangle_L |\uparrow\rangle_R \, {}_L\langle\downarrow| \, {}_R\langle\uparrow|$$

$$\tilde{\rho} = {}_L\langle\uparrow|\rho|\uparrow_L\rangle + {}_L\langle\downarrow|\rho|\downarrow\rangle_L$$ (Trace over both L states)

$$= {}_L\langle\uparrow| \left(\tfrac{1}{2}|\uparrow\rangle_L |\downarrow\rangle_R \, {}_R\langle\downarrow| \, {}_L\langle\uparrow|\right) |\uparrow\rangle_L + 0 - 0 + \dots$$
$$+ \, {}_L\langle\downarrow| \left(\tfrac{1}{2}|\downarrow\rangle_L |\uparrow\rangle_R \, {}_R\langle\uparrow| \, {}_L\langle\downarrow| \, |\downarrow\rangle_L\right.$$
$$= \tfrac{1}{2} |\downarrow\rangle_R \, {}_R\langle\downarrow| + \tfrac{1}{2} |\uparrow\rangle_R \, {}_R\langle\uparrow|$$

= 50/50 mixture of up & down right atom spin

Unpolarized beam!

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & \cancel{-1/2} & 0 \\ 0 & \cancel{-1/2} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Coherence stored in left atom

$$\tilde{\rho} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Decoherence destroys off-diagonal terms

- Observations destroy off-diagonal coherence
- Departing information destroys off-diagonal coherence ~~too~~ (Feynman/Wheeler) 2nd atom/photon off to ∞
- Interaction w/ ~~...~~ environment can also destroy coherence (Departing into 10^{23} dimensions...)

Observations

⇒ Remove coherence between eigenstates

Define $\psi_{\alpha\beta} = \langle O_\alpha | \psi \rangle \langle \psi | O_\beta \rangle$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \psi_{00} & \psi_{01} & \psi_{02} \\ \psi_{10} & \psi_{11} & \cdots \\ \psi_{20} & \vdots & \end{pmatrix} \xrightarrow{\text{observe}} \begin{pmatrix} \psi_{00} & 0 & 0 \\ 0 & \psi_{11} & 0 \\ 0 & & \psi_{22} \end{pmatrix}$$

Observation of O ≡ Destruction of coherence ~~between~~ (in eigenstate basis

Idea: L ~~deuterium atom~~ He ion hits 'macroscopic object' (detector), entangles with it. {eats electron opposite spin, net spin of macro object entangled}

Macro-objects can't interfere with themselves: $\cancel{\langle \uparrow | \nearrow \rangle}$

$$\langle \uparrow | O | \nearrow \rangle = 0$$

For any observable, perturbation, time evolution O. (Different Hilbert spaces)

[Coherence demands two paths leading to exactly same final state. Needle + motion leaves micro-scratches, air currents, ~~[illegible]~~ photons leaving through outer space (c.f. Feynman Wheeler again)

→ ~~can never interfere~~.

$|\uparrow\rangle_R$ & $|\downarrow\rangle_R$ can never interfere again.

$$\cancel{|\uparrow_R\rangle = |\textcircled{\nwarrow}\rangle|\uparrow_R\rangle}$$

$$\Psi = \tfrac{1}{\sqrt{2}}\left(|\textcircled{\nwarrow}\rangle|\uparrow_R\rangle + |\textcircled{\nearrow}\rangle|\downarrow_R\rangle\right)$$

$$\langle\psi|O|\psi\rangle = \tfrac{1}{2}\langle\uparrow_R|\langle\textcircled{\nwarrow}|\,O\,|\textcircled{\nwarrow}\rangle|\uparrow_R\rangle$$
$$+\ \tfrac{1}{2}\langle\downarrow_R|\langle\textcircled{\nearrow}|\,O\,|\textcircled{\nearrow}\rangle|\downarrow_R\rangle$$
$$+\ \cancel{\text{zero}}\ \underbrace{\langle\textcircled{\nwarrow}|O|\textcircled{\nearrow}\rangle}_{\text{zero}} + \text{zero}$$

$\equiv$ Incoherent ~~superposition~~ mixture!

prob ½ $|\textcircled{\nwarrow}\rangle|\uparrow\rangle$

prob ½ $|\textcircled{\nearrow}\downarrow\rangle$

We shall ~~see~~ ~~state~~ study this "different Hilbert spaces" ~~stuff~~ stuff later on course.

- Contact w/ solid not enough.
- Contact w/ metal can be enough
- Overlap catastrophe, infrared divergence, Macroscopic quantum tunneling, Kondo, X-ray edge, ...