'Second quantization' - commutation etc.

Last lecture:
  * Fock space = occupation number representation
  * Annihilation operator ai, reduces number in state i
  * Creation operator increases number
  * Multiply by sqrt(Ni) makes single-particle operators simple

$$\langle N_k - 1 | a_k | N_k \rangle = \sqrt{N_k} \qquad \langle N_i | a_i^+ | N_i - 1 \rangle = \sqrt{N_i}$$

$$V = \sum_i V(x_i) = \sum_{ik} V_{ik} \, a_i^+ a_k \qquad V_{ik} = \int \mathcal{U}_i^*(x) \, V(x) \, \mathcal{U}_k(x) \, dk$$

$$U = \sum_{i<j} U(x_i, x_j) = \sum_{ik\ell m} U_{ik\ell m} \, a_i^+ a_k^+ a_\ell a_m \quad \cdots$$

Basis for number counting

Pick basis of position states.
  * $|0\rangle$ = state with no particles

Note: a_x^+, $a_{x'}$ often written $\hat{\mathcal{U}}^+(x)$, $\hat{\mathcal{U}}(x')$

$$\bullet \; |x\rangle = a_x^+ |0\rangle \;\; \Rightarrow \;\; |\mathcal{U}\rangle = \int \mathcal{U}(x) \, a_x^+ |0\rangle$$

$y_1 = x_2 \quad y_2 = x_1$

$$\bullet \; |\mathcal{U}(x_1, x_2)\rangle = \int dx_1 \, dx_2 \, \mathcal{U}(x_1, x_2) a_{x_1}^+ a_{x_2}^+ |0\rangle = \int dy_2 \, dy_1 \, \mathcal{U}(x_2, y_1) a_{y_2}^+ a_{y_1}^+ (g)$$

Bosons $\mathcal{U}(x_1, x_2) = \mathcal{U}(x_2, x_1) \Rightarrow a_{x_1}^+ a_{x_2}^+ = a_{x_2}^+ a_{x_1}^+ \Rightarrow [a_i^+, a_j^+] = 0$

Fermions $\mathcal{U}(x_1, x_2) = -\mathcal{U}(x_2, x_1) \Rightarrow a_{x_1}^+ a_{x_2}^+ = -a_{x_2}^+ a_{x_1}^+$

$$\Rightarrow \underbrace{\{a_i^+, a_j^+\}}_{\text{Anticommutator}} = a_i^+ a_j^+ + a_j^+ a_i^+ = 0$$

Fermions a_i^+, a_j usually called c_i^+, c_j

Hermitian Conjugate $\Rightarrow [a_i, a_j] = \{c_i, c_j\} = 0$

Q: Does $a_i^+ a_i$ change N_i? What is it?

A: $a_i^+ a_i |N_i\rangle = a_i^+ (\sqrt{N_i} |N_i - 1\rangle) = \sqrt{N_i} \sqrt{N_i} |N_i\rangle$

$a_i^+ a_i = N_i$ <u>Number Operator</u>

Remember, non-interacting particles energy ε_i

$$\mathcal{H} = \sum_i \varepsilon_i a_i^+ a_i = \sum_i \varepsilon_i N_i \quad \text{(makes sense)}$$

$\overbrace{\qquad}$ True for bosons

Q: What is $a_i a_i^+$? What is $[a_i, a_i^+]$?

A: $a_i a_i^+ \underbrace{|N_i\rangle}_{\substack{N_k-1 \\ \text{in formula}}} = a_i \underbrace{\sqrt{N_i+1}}_{N_k} |N_i\rangle = (N_i+1)|N_i\rangle$

$[a_i, a_i^+] = a_i a_i^+ - a_i^+ a_i = (N_i+1) - N_i = 1$

$\left. \begin{array}{l} [a_i, a_j^+] = \delta_{ij}, \quad \{c_i, c_j^+\} = \delta_{ij} \\[4pt] [a_i, a_j] = 0 \quad \{c_i, c_j\} = 0 \\[4pt] [a_i^+, a_j^+] = 0 \quad \{c_i^+, c_j^+\} = 0 \end{array} \right\}$ Commutation Relations

$\left\{ c_i, c_j^+ \right\} = \delta_i^j$ without

* Hamiltonians almost always in second quantized form
* Gregarious bosons exercise:

$$a_r^+ = \widehat{\psi^+(\vec{r})} = \sum \underbrace{\langle \vec{r}|\psi_i\rangle}_{\frac{1}{\sqrt{M}}} a_i^+$$

M lowest states

$$a_r^+ (a_o^+)^{N_o} |0\rangle = \sum_{i=0}^{m-1} \frac{1}{\sqrt{M}} a_i^+ |N_o\rangle$$

$$= \frac{1}{\sqrt{M}} \left\{ \underbrace{\sqrt{N_o+1}}_{\substack{\text{Probability up by} \\ N_o+1 \,!}} |N_o+1\rangle + \sum_{i=1}^{m-1} |N_o, 0, 0, \dots, N_i=1, 0, \dots\rangle \right\}$$

Bosons like to collect in same state:
  Lasers, superfluids

$\left[\text{View exercise as new derivation} \quad a_i^+|N_i\rangle = \sqrt{N_i+1}\,|N_i\rangle \right]$

* Off-diagonal long-range order extra credit exercise:
  apply a, adag to ground state of many-particle soup

$$\langle \Psi | a^+(r') a(r) | \Psi \rangle = \text{'reduced' density matrix}$$

(tracing over all but one particle)