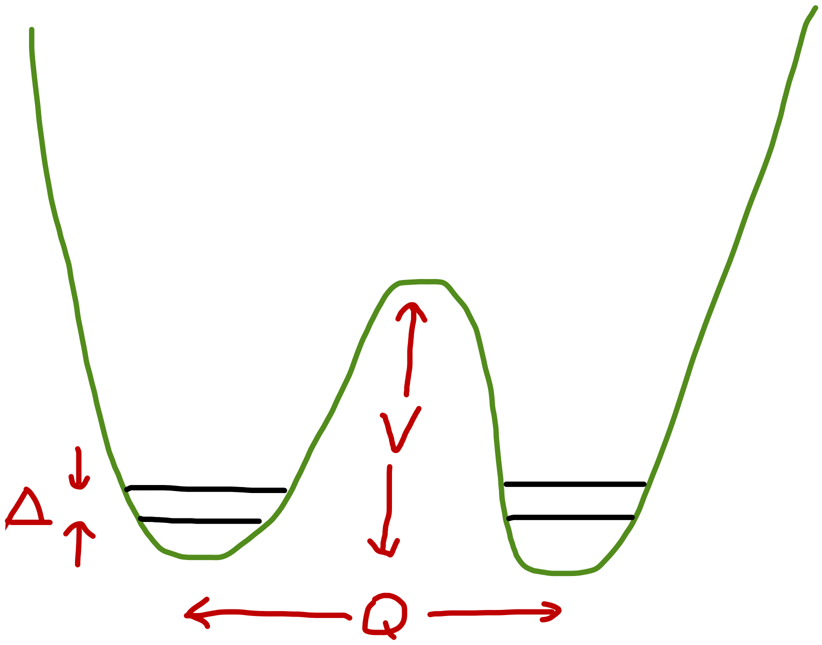
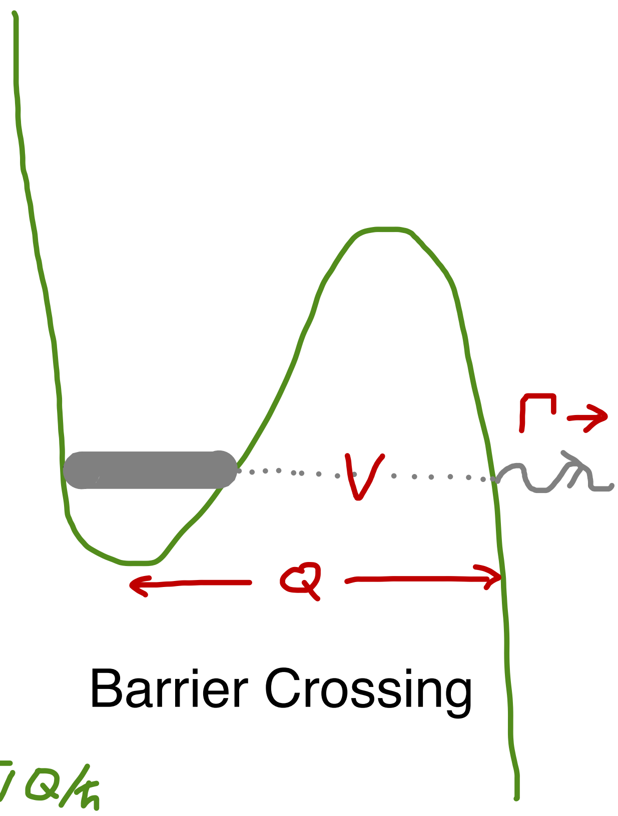
Instantons, Quantum Tunneling, and WKB



Double Well



Barrier Crossing

$$\text{WKB: } \Delta \sim e^{-\sqrt{2MV}\,Q/\hbar},$$
$$\Gamma \sim \Delta^2$$

How to use path integrals? Rotate to imaginary time!

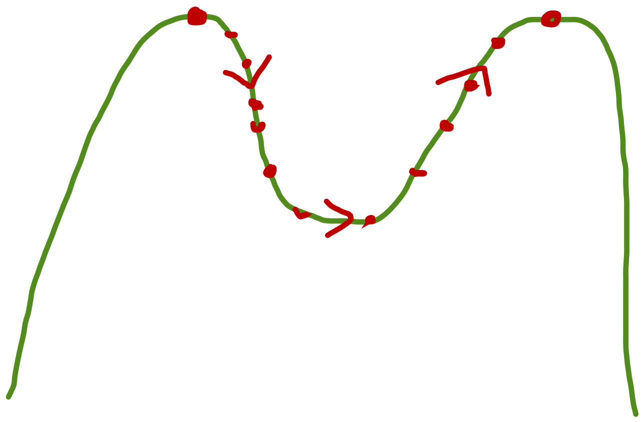
$$\langle x', t' | x_0, t_0 \rangle = \int \mathcal{D}[x(t)]\; e^{\frac{i}{\hbar}\int \frac{1}{2} m \dot{x}^2 - V(x)\, dt}$$

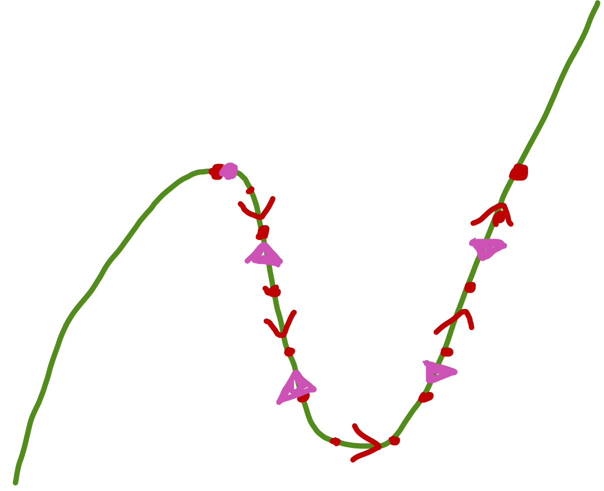
$$\tau = it; \quad -i\,d\tau = dt; \quad \dot{x}^2 = (dx/dt)^2 = -\left(\frac{dx}{d\tau}\right)^2 \equiv x'^2$$

$$= \int \mathcal{D}[x(\tau)]\; \exp\left[-\frac{1}{\hbar}\int \frac{1}{2} m x'^2 + V(x)\, d\tau\right]$$

* Analytic continuation???
* Physics, not math. Gives different information!
* No oscillations! Biggest minimizes Euclidean action:

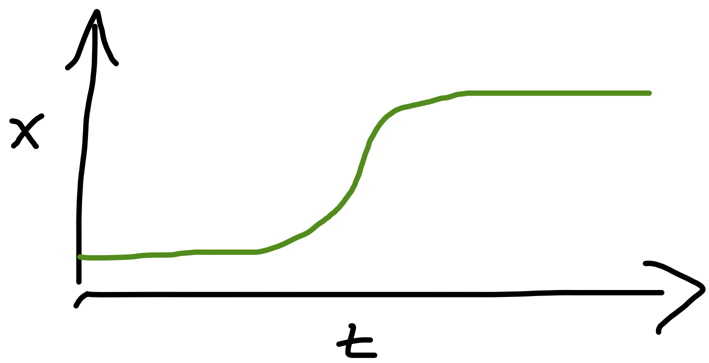
$$S_E = \int \frac{1}{2} m x'^2 + V(x)\, d\tau = \underline{\text{Inverted}} \text{ potential}$$



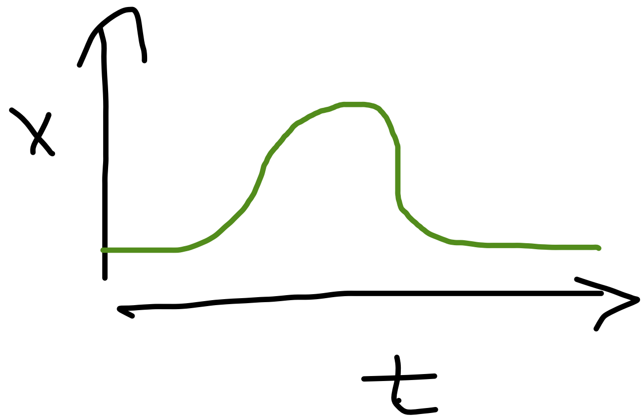


Instanton: Falls in, rolls up other side

Barrier crossing: bounces off turning point



Soliton in time -> Instanton

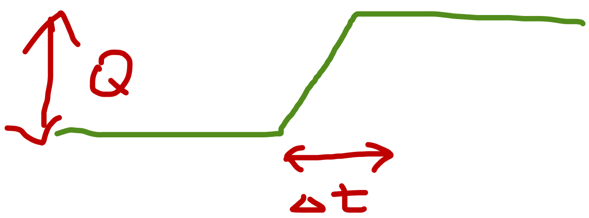


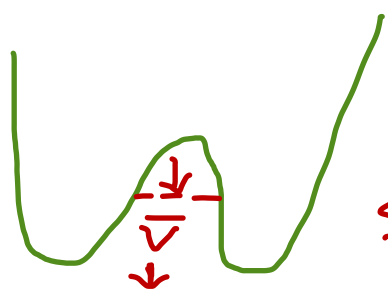
Instanton Bounce
Crosses barrier twice

$\Gamma \sim \Delta^2$

How to get WKB?

(1) Variational bound: Ramp





$S_E \leq \frac{1}{2} M \, Q^2/\Delta t^2 \, \Delta t + \bar{V} \Delta t$

Minimize wrt Δt: $-\frac{1}{2} M Q^2/\Delta t^2 + \bar{V} = 0$

$\Delta t = \sqrt{\frac{1}{2} m Q^2 / \bar{V}}$

$S_E \leq \sqrt{\frac{1}{2} M Q^2 \bar{V}} + \sqrt{\frac{1}{2} m Q^2 \bar{V}} = \sqrt{2 m \bar{V}} \, Q$

$\Delta = \underbrace{\{\hbar \omega_0\}}_{\text{Fluctuations}} e^{-\sqrt{2 m \bar{V}} \, Q/\hbar}$

How to get WKB? Great trick: 'Energy' conservation

Define V(bottom)=0. Path starts at bottom with xdot=0, so zero 'E'

$$\text{'}E\text{'} = \tfrac{1}{2} m \dot{x}^2 - V(x) = 0 \quad \rightarrow \quad \dot{x} = \sqrt{2V(x)/m}$$

$$\rightarrow S_E = \int \left(\tfrac{1}{2} m \underbrace{\dot{x}} + V(x) / \underbrace{\dot{x}}\right) \underbrace{\left(\frac{dx}{dt} dt\right)}_{dx}$$

$\dot{x} = \sqrt{2V(x)/m}$; $\frac{dx}{dt} dt = dx$

$$= \int \tfrac{1}{2} m \sqrt{\frac{2V(x)}{m}} + \frac{V(x)}{\sqrt{2V(x)/M}} \, dx = \int dx \; 2\sqrt{\frac{mV(x)}{\hbar}}$$

$$S_E = \int dx \sqrt{2mV(x)}$$

Tunneling matrix element

$$\Delta = (\text{prefactor}) \exp\left(-\int \sqrt{2mV(x)}\, dx/\hbar\right)$$

Barrier crossing rate

$$\Gamma = (\text{prefactor}) \exp\left(-\underbrace{2}_{\text{Crosses Barrier Twice}} \int \sqrt{2mV(x)}\, dx/\hbar\right)$$

Prefactor?
- Group project #2: Monte Carlo?
- WKB: Matching to ground states
- Wronskians ...

See "The Uses of Instantons", by Sydney Coleman (reprinted in his "Aspects of Symmetry")