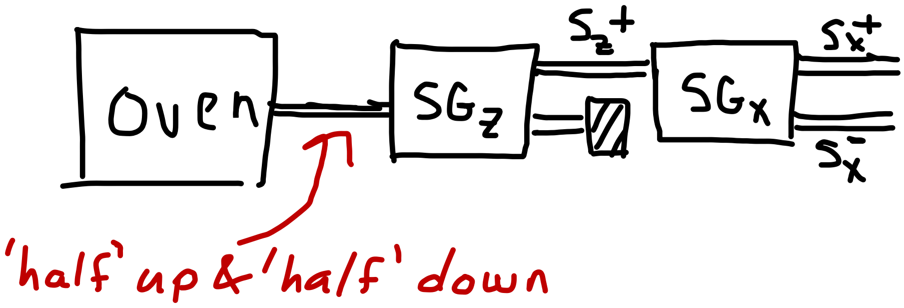
Density Matrices

Describe 'mixed' spin state of Stern Gerlach oven beam



What is an entangled beam once it decoheres?

Stop using wavefunctions. Use 'density matrices'.
(Matrix has room for mixtures of several quantum states)

Wavefunction $\psi(x) = |\psi\rangle$

becomes 'pure state' density matrix $|\psi\rangle\langle\psi| = \rho$

Projection operator, taking component of $|\varphi\rangle$ along $|\psi\rangle$

$$\rho\,|\varphi\rangle = |\psi\rangle \underbrace{\langle\psi|\varphi\rangle}_{\text{Amplitude along } |\psi\rangle}$$

Contains all physical information?

$$\psi(x) = \langle x|\psi\rangle, \text{ so } \langle x|\rho|x'\rangle = \langle x|\psi\rangle\langle\psi|x'\rangle$$
$$= \psi(x)\,\psi^*(x')$$

Fix x' s.t. $\psi^*(x') = A \neq 0$.

$\langle x|\rho|x'\rangle = A\,\psi(x)$ (unknown A).

Normalize, $\int dx\, |\langle x|\rho|x'\rangle|^2 = \int dx\, |A|^2 |\psi(x)|^2 = |A|^2$

$$\frac{\langle x|\rho|x'\rangle}{\sqrt{\int |\langle x|\rho|x'\rangle|^2}} = e^{i\varphi}\psi(x), \text{ unknown phase } \varphi.$$

Overall phase of WF is not measurable (only phase differences):
Pure state density matrix has all info about state

2nd level of probability: ensemble has probabilities p_n of being in many orthogonal quantum states psi_n:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

Is QM awkward with density matrices? Slightly...

Time evolution: $\frac{\partial \rho}{\partial t} = \sum_n p_n \left(\underbrace{\frac{\partial |\psi_n\rangle}{\partial t}}_{\frac{1}{i\hbar} H|\psi_n\rangle} \langle\psi_n| + |\psi_n\rangle \underbrace{\frac{\partial \langle\psi_n|}{\partial t}}_{-\frac{1}{i\hbar}\langle\psi_n|H} \right)$

$$= \frac{1}{i\hbar}(H\rho - \rho H) = \frac{1}{i\hbar}[H,\rho] \qquad \frac{\partial \psi_n^*}{\partial t} = \left(\frac{\partial \psi_n}{\partial t}\right)^*$$

Note: opposite of Heisenberg picture

$$\frac{\partial O}{\partial t} = \frac{1}{i\hbar}[O,H]$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H,\rho] = -\frac{1}{i\hbar}[\rho,H]$$

Operator Expectation: $\langle\psi|O|\psi\rangle = Tr(O\rho)$

$Tr(M) = \sum M_{ii}$, independent of basis

Why? Try $|x\rangle$ basis

$$Tr(O\rho) = \sum_n p_n Tr(O|\psi\rangle\langle\psi|)$$

$$= \sum_n p_n \int dx \underbrace{\langle x|O|\psi\rangle}_{(O\psi)(x)} \underbrace{\langle\psi|x\rangle}_{\psi^*(x)}$$

$$= \sum_n p_n \int dx\, \psi^*(x)\, O\, \psi(x)\, dx$$

$$= \sum_n p_n \langle\psi_n|O|\psi_n\rangle$$

Observation by O: Weird! Not Schrodinger eqn.

Pure state $|\psi\rangle \to$ Mixture $|o_\alpha\rangle$ *Just 'before' measurement*

prob $p_\alpha = |\langle o_\alpha|\psi\rangle|^2$

Pure State:

$$\rho = |\psi\rangle\langle\psi| \text{ in basis } |o_\alpha\rangle$$

$$\rho = \mathbb{1}\,\rho\,\mathbb{1} = \Big(\sum_\alpha |o_\alpha\rangle\langle o_\alpha|\Big)\,\rho\,\Big(\sum_\beta |o_\beta\rangle\langle o_\beta|\Big)$$

$$= \sum_{\alpha,\beta} |o_\alpha\rangle\langle o_\alpha|\psi\rangle\langle\psi|o_\beta\rangle\langle o_\beta|$$

$$\rho_{\alpha\beta} = \langle o_\alpha|\psi\rangle\,\langle o_\beta|\psi\rangle^*$$

After Observation, Mixture:

$$\rho' = \sum p_\alpha |o_\alpha\rangle\langle o_\alpha|$$

$$= \sum |\langle o_\alpha|\psi\rangle|^2 |o_\alpha\rangle\langle o_\alpha|$$

$$\rho'_{\alpha\alpha} = \langle o_\alpha|\psi\rangle\langle o_\alpha|\psi\rangle^* = \rho_{\alpha\alpha}$$

$$\rho'_{\alpha\beta} = 0 \quad \alpha \neq \beta \text{ !}$$

Observation by 'classical' instrument destroys off-diagonal 'coherence'!

Back to S-G beam! Density matrix for singlet pure state:

$$|s\rangle\langle s| = \rho =$$

$$= \frac{1}{2}\left(|\uparrow\rangle_L|\downarrow\rangle_R - |\downarrow\rangle_L|\uparrow\rangle_R\right)\left({}_L\langle\uparrow|\,{}_R\langle\downarrow| - {}_L\langle\downarrow|\,{}_R\langle\uparrow|\right)$$

$$= \frac{1}{2}|\uparrow\rangle_L|\downarrow\rangle_R\,{}_L\langle\uparrow|\,{}_R\langle\downarrow| - \frac{1}{2}|\uparrow\rangle_L|\downarrow\rangle_R\,{}_L\langle\downarrow|\,{}_R\langle\uparrow|$$
$$- \frac{1}{2}|\downarrow\rangle_L|\uparrow\rangle_R\,{}_L\langle\uparrow|\,{}_R\langle\downarrow|$$
$$+ \frac{1}{2}|\downarrow\rangle_L|\uparrow\rangle_R\,{}_L\langle\downarrow|\,{}_R\langle\uparrow|$$

In the basis $\begin{pmatrix}\uparrow\uparrow\\ \uparrow\downarrow\\ \downarrow\uparrow\\ \downarrow\downarrow\end{pmatrix}$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Coherence between L&R in off-diagonal -1/2

As atoms separate, exposed to environment, coherence lost
Or, just 'forget' about L atoms!

Implement as Partial Trace...

Agree never to measure atoms left behind in oven (L-atoms).
All observables expandable in R basis:

$$O = \sum_{\substack{r_\alpha \in \uparrow_R, \uparrow_L \\ r_\beta \in}} O_{\alpha\beta}\, |r_\alpha\rangle_R\, {}_R\langle r_\alpha|$$

Expectation value

$$\mathrm{Tr}(O\rho) = \sum_{r_\alpha} \sum_{\ell_\mu} {}_R\langle r_\alpha|\, {}_L\langle \ell_\mu|\, O\rho\, |\ell_\mu\rangle_L\, |r_\alpha\rangle_R$$

$$= \sum_{r_\alpha} \langle r_\alpha|\, O \underbrace{\left(\sum_{\ell_\mu} {}_L\langle \ell_\mu|\rho|\ell_\nu\rangle_L\right)}_{\tilde{\rho} = \text{Partial Trace over Forgotten Info}} |r_\alpha\rangle$$

Partial Trace

$$\tilde{\rho} = \sum_{\mu} \langle \ell_\mu | \rho | \ell_\nu \rangle$$

Contains all information necessary to predict R-atom observables.

Beam from S-G oven?

$$\rho = \tfrac{1}{2} \Big(|\uparrow\rangle_L |\downarrow\rangle_R \, {}_L\langle\uparrow| \, {}_R\langle\downarrow| - \tfrac{1}{2} |\uparrow\rangle_L |\downarrow\rangle_R \, {}_L\langle\downarrow| \, {}_R\langle\uparrow| \Big)$$

$$- \tfrac{1}{2} |\downarrow\rangle_L |\uparrow\rangle_R \, {}_L\langle\uparrow| \, {}_R\langle\downarrow| + \tfrac{1}{2} |\downarrow\rangle_L |\uparrow\rangle_R \, {}_L\langle\downarrow| \, {}_R\langle\uparrow|$$

$$\tilde{\rho} = {}_L\langle\uparrow| \rho |\uparrow\rangle_L + {}_L\langle\downarrow| \rho |\downarrow\rangle_L$$ (Trace over both L states)

$$= {}_L\langle\uparrow| \Big(\tfrac{1}{2} |\uparrow\rangle_L |\downarrow\rangle_R \, {}_R\langle\downarrow| \, {}_L\langle\uparrow| \Big) |\uparrow\rangle_L + 0 - 0 + \ldots$$

$$+ {}_L\langle\downarrow| \Big(\tfrac{1}{2} |\downarrow\rangle_L |\uparrow\rangle_R \, {}_R\langle\uparrow| \, {}_L\langle\downarrow| \Big) |\downarrow\rangle_L$$

$$= \tfrac{1}{2} |\downarrow\rangle_R \, {}_R\langle\downarrow| + \tfrac{1}{2} |\uparrow\rangle_R \, {}_R\langle\uparrow|$$

= 50/50 mixture of up & down right atom spin

Unpolarized beam!

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & \cancel{-1/2} & 0 \\ 0 & \cancel{-1/2} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Coherence stored in left atom

$$\tilde{\rho} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Decoherence destroys off-diagonal terms

Observations destroy off-diagonal coherence

Departing information destroys off-diagonal coherence too