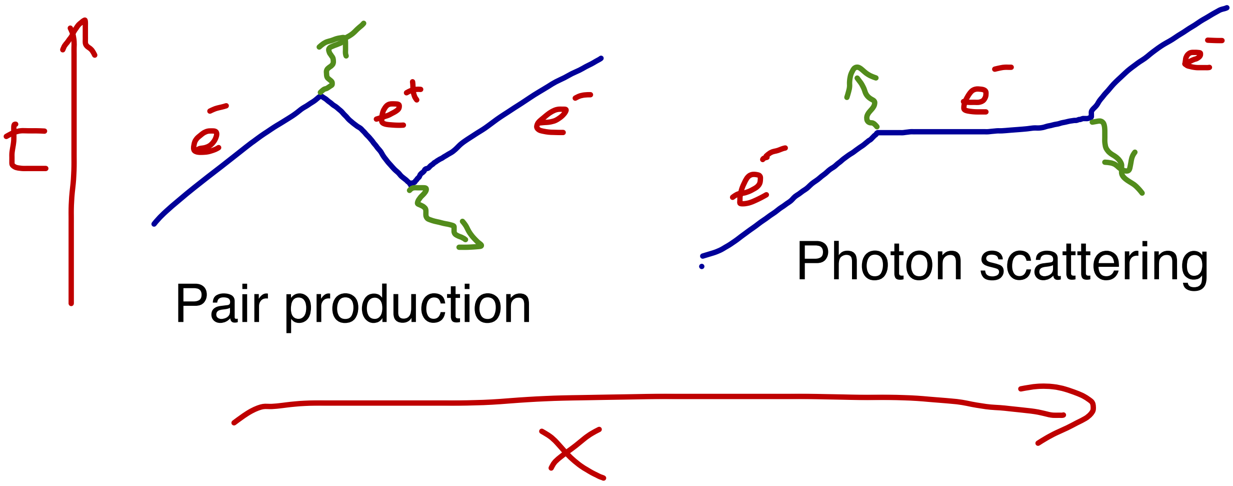
Bosons and Fermions

Quantum identical particles are truly the same.



Wheeler:
Positron = same electron backward in time.

$$\Psi(\vec{r}_1,\vec{r}_2) = \underbrace{e^{i\varphi}}_{\pm 1}\,\Psi(\vec{r}_2,\vec{r}_1) = \underbrace{e^{2i\varphi}}_{1}\,\Psi(\vec{r}_1,\vec{r}_2)$$

(in 3D. In 2D more subtle: see Anyons, ps #4)

Bosons: +1
mesons, He4, phonons, photons, gluons, W+-, Z, gravitons

Fermions: -1
electrons, protons, neutrons, neutrinos, quarks.

Spin statistics theorem (relativistic QM):
integer spin = boson,
half-integer = Fermion

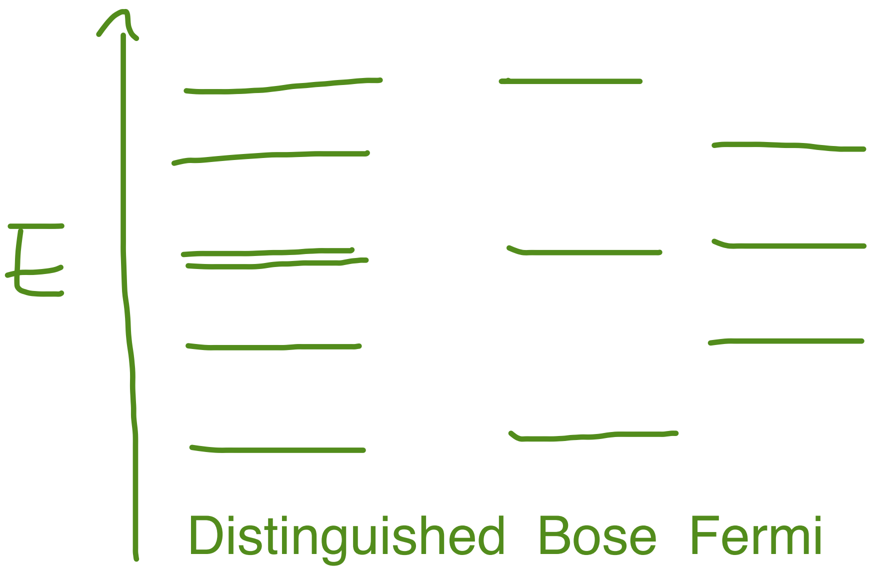
Q: Is a hydrogen atom a fermion or a boson?
Composites w/even # fermions = bosons

Many particles?
Permutation P = {P1,...,P_N} reordering integers {1,2,...,N}.
sigma(P)=+1 if P even permutation (net even number of swaps), sigma(P) = -1 for odd permutation

Bosons: $\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \Psi(\vec{r}_2, \vec{r}_1, \ldots, \vec{r}_N) = \Psi(\vec{r}_{P_1}, \vec{r}_{P_2}, \ldots, \vec{r}_{P_N})$

Fermions: $\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = -\Psi(\vec{r}_2, \vec{r}_1, \ldots, \vec{r}_N) = \sigma(P)\Psi(\vec{r}_{P_1}, \ldots, \vec{r}_{P_N})$

Fermion / bose wavefunctions confined to odd / even subspace of Hilbert space.



Q: Given eigenstates for a (distinguishable particle) Hamiltonian, how to form Bose & Fermi eigenstates?

A: Since [H,P] = 0, [H,sigma(P)]=0 too: eigenstates can be chosen to be either even or odd under permutations. Degenerate states with mixed symmetry can be symmetrized (Boson):

$$\Psi_{sym} = (\text{normalization}) \sum_P \Phi(\vec{r}_{P_1}, \ldots, \vec{r}_{P_N})$$

or antisymmetrized (Fermion):

$$\Psi_{asym} = (\text{norm}) \sum_P \sigma(P)\, \Phi(\vec{r}_{P_1} \ldots \vec{r}_{P_N})$$

If non-zero, some of energy-E eigenstates energy E.

Groups and Permutation Groups

A group G is a set {g1,...,g_N} closed under a product * that
is Associative: (g_i * g_j) * g_k = g_i * (g_j * g_k)
has an Identity e*g_i = g_i = g_i * e
has an Inverse for each element: g_i * (g_i^-1) = e

Q: Which are groups?
Integers Z under addition
Integers Z under multiplication
Rotations in SO(3)
SU(2) (spin 1/2 rotations)
Permutations of 5 particles

Q: How many elements are there in the permutation group of 5 particles?

Q: In the permutation group of three particles
(e=123,132,213,231,312,321)
which are even and which are odd? Do the even permutations form a subgroup?

What about spin? For light atoms in zero field, the Hamiltonian is spin independent, so the wavefunction factors:

$$\text{He: } \Psi(\vec{r}_1, s_1, \vec{r}_2, s_2) = \psi(\vec{r}_1, \vec{r}_2)\,\chi(s_1, s_2)$$

(Electrons near heavy nuclei are relativistic, leading to spin-orbit scattering coupling L and S)...

Ground State $\chi = (\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$ antisymmetric, $S=0$

$\underbrace{\psi(\vec{r}_1, \vec{r}_2)}$ symmetric $E = -78.95\,\text{eV}$

Not $\psi_{1s}(\vec{r}_1)\psi_{1s}(\vec{r}_2)$, 'independent electron approximation'

Excited triplet state: $E = -59.13\,\text{eV}$

$$\chi^* = \begin{Bmatrix} \uparrow\uparrow \\ (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2} \\ \downarrow\downarrow \end{Bmatrix}, \quad \psi^* \text{ antisymmetric}$$

$$\text{Li (3 electrons): } \psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)\,\chi(s_1, s_2, s_3)$$

Q: Can the spatial psi be totally symmetric?

A: Can't write totally antisymmetric spin WF χ with 3 particles

E.g. $\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow = 0$

Q: Is the ground state spatial psi totally antisymmetric?

A: No. 'Independent electron' picture $\underbrace{1s^2}_{\text{Like He core}}, \underbrace{2s^1}_{\text{'Valence' electron}}$

Spin singlet core, total spin ½

(Spin 3/2 = totally symmetric)

Both spatial and spin WF have mixed symmetry.
Young Tableau, etc.

Shortcut: Treat 'up' and 'down' as two different kinds of particles (antisymmetrize separately). Spin handles rest of antisymmetrization. (Not sure this works when large spin-orbit coupling, Z >~30.)

Converse: "Isospin" in nuclear physics. Treat p and n as identical particles, with isospin = +1/2 and -1/2. Electromagentism and quark mass differences give 'small' symmetry breaking.