Rotate vector about $\hat{z}$ by angle ϕ:

$V \to RV$

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

spinor

Rotate spin $\begin{pmatrix} \alpha_\uparrow \\ \alpha_\downarrow \end{pmatrix} = \alpha_\uparrow |\uparrow\rangle + \alpha_\downarrow |\downarrow\rangle$ about $\hat{z}$ by ϕ?

Be very careful!

weird answer

Q: Does $|\uparrow\rangle \xrightarrow{R_\phi} |\uparrow\rangle$?

A: Only up to phase. How to guess phase?

Q: If $|\uparrow\rangle \xrightarrow{\phi} e^{i\Phi} |\uparrow\rangle$,

what is $|\uparrow\rangle$ after 2ϕ rotation?

A: $e^{i2\Phi}$, so $|\uparrow\rangle \to e^{ix\varphi} |\uparrow\rangle$ for some x.

Infinitesimal rotation about $\hat{z}$ $|\uparrow\rangle \to (1 + ix\,d\varphi)|\uparrow\rangle$

Q: Guess: $|\downarrow\rangle \xrightarrow{\phi} e^{?} |\downarrow\rangle$.

Also $e^{ix\varphi}$? No, then rotation doesn't change state.

$e^{-ix\varphi}$? ✓

How to find x?

Know $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ has eigenstate $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = |\uparrow_x\rangle$

Surely $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $|\uparrow_y\rangle = \begin{pmatrix}1\\ -i\end{pmatrix}/\sqrt{2}$

or $= \begin{pmatrix}\sqrt{i}\\ 1/\sqrt{i}\end{pmatrix}/\sqrt{2}$

Know $|\uparrow_x\rangle \xrightarrow{\phi = \pi/2} |\uparrow_y\rangle$, use to find x

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \xrightarrow{\phi = \pi/2} \begin{pmatrix}e^{ix\phi}\\ e^{-ix\phi}\end{pmatrix} = \begin{pmatrix}\sqrt{i}\\ -\sqrt{i}\end{pmatrix}$$

Q: What is x?

A: $x = 1/2$

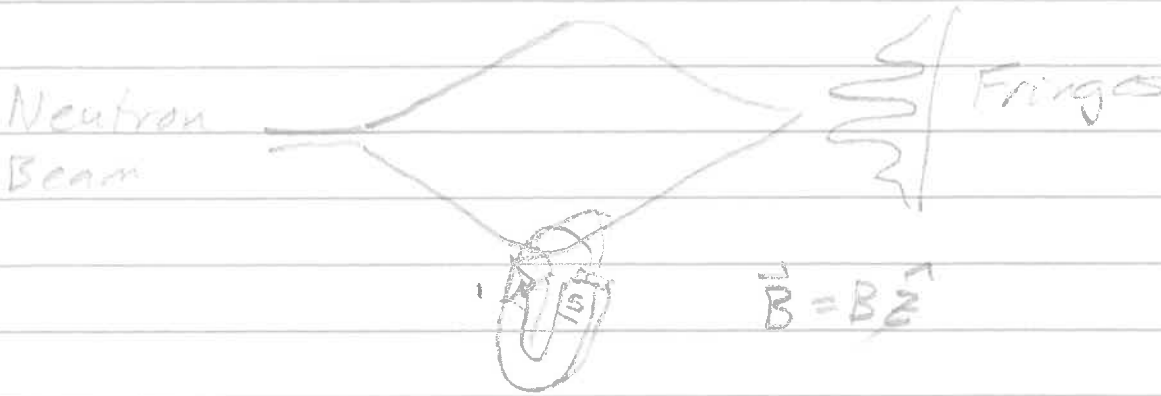
Q: What happens to $\begin{pmatrix}\alpha_\uparrow\\ \alpha_\downarrow\end{pmatrix}$ after $360°$ rotation about $\vec{z}$?

A: $-\begin{pmatrix}\alpha_\uparrow\\ \alpha_\downarrow\end{pmatrix}$!

Check: $90°$ physical rotation $|\uparrow\rangle \to \sqrt{i}\,|\uparrow\rangle$

$360°$ $|\uparrow\rangle \to (\sqrt{i})^4\,|\uparrow\rangle = i^2\,|\uparrow\rangle = -|\uparrow\rangle$ ✓

$\boxed{D_z(\phi) = \begin{pmatrix}e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{pmatrix}}$ rotation of spinor

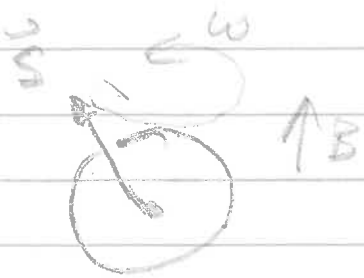


Hamiltonian $\mathcal{H} = -\vec{\mu}\cdot\vec{B}$ $\qquad |\mu| = g_n \frac{e\hbar}{2m_p c}$ $\quad g_n \simeq -1.91$

$$\vec{\mu} = g_n \frac{e}{m_p c}\vec{S}$$

$$\mathcal{H} = -g_n \frac{e}{m_p c}\vec{S}\cdot\vec{B}$$

$$= -g_n \frac{eB}{m_p c} S_z = \begin{pmatrix} -g_n \frac{eB\hbar}{2m_p c} & 0 \\ 0 & g_n \frac{eB\hbar}{2m_p c} \end{pmatrix}$$



"Gyroscopic" precession frequency

$$\omega = \frac{g_n e B}{m_p c}$$

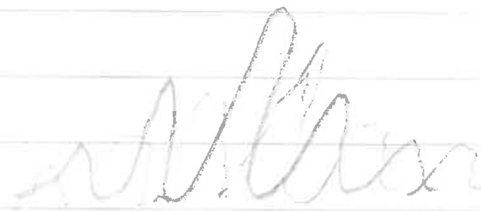
$$\mathcal{H} = \begin{pmatrix} -\frac{\hbar\omega}{2} & 0 \\ 0 & \frac{\hbar\omega}{2} \end{pmatrix}$$

Q! What is time evolution $\mathcal{U}(t)$? Is it a rotation $\mathcal{D}_z(\phi)$? what is ϕ?

A! $\mathcal{U}(t) = e^{-\frac{i\mathcal{H}t}{\hbar}} = \begin{pmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{pmatrix}$

$= \mathcal{D}_z(\omega t)$

Neutron beam
after $\omega t = 0$
$(B = 0)$



after $\omega t = 2\pi$

Rotate by $2\pi \rightarrow \Psi_n \rightarrow -\Psi_n$!

Feynman plate trick