SU(2) and spinor rotations

$$\mathcal{D}_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} = e^{-\frac{i}{\hbar}\underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}}_{S_z}\phi}$$

$$= e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{z}\phi} \qquad \vec{S} = (S_x, S_y, S_z)$$

$$= \frac{\hbar}{2}\underbrace{(\sigma_x, \sigma_y, \sigma_z)}_{\text{pauli matrices}} = \frac{\hbar}{2}\vec{\sigma}$$

- Any 3D rotation has axis $\hat{n}$ invariant, angle ϕ

$\hat{n}\phi$ = ~~Rodrigues~~ Axis-angle coord (vs. Euler angles)

$$\vec{\sigma}\cdot\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$$

$$\mathcal{D}_n(\phi) = e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\phi}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} n_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} n_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_z$$

$$\underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}}_{\text{spinor}} \xrightarrow{\hat{n},\phi} \mathcal{D}_{\hat{n}}(\phi)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}}_{\text{Note: Hermitian}} = \left(\frac{2}{\hbar}\vec{S}\cdot\hat{n}\right)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_z^2 = \mathbb{1} \qquad \text{Rotate } \vec{z} \to \hat{n}$$

$$(\sigma\cdot\hat{n})^2 = \mathbb{1} \qquad \left\{\text{Check: } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\}$$

$$(\sigma\cdot\hat{n})^n = \begin{cases} \mathbb{1} & n \text{ even} \\ \sigma\cdot n & n \text{ odd} \end{cases}$$

$$\Rightarrow e^{iA\,\sigma\cdot n} = \sum \frac{(iA)^n}{n!}(\sigma\cdot n)^n = \underbrace{\sum_{\ell=0} (-1)^\ell \frac{A^{2\ell}}{2\ell!}}_{\cos A}\mathbb{1} + i\underbrace{(-1)^\ell \frac{A^{2\ell+1}}{(2\ell+1)!}}_{\sin A}(\sigma\cdot\hat{n})$$

$$= \mathbb{1}\cos A + i\,\vec{\sigma}\cdot\hat{n}\sin A$$

$$-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\phi = -\frac{i}{2}\sigma\cdot n\,\phi \qquad A = -\frac{\phi}{2}$$

$$\mathcal{D}_n(\phi) = e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\phi} = \cos(-\phi/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(-\phi/2)\overbrace{\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}}^{i\,\sigma\cdot n}$$

$$= \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \begin{aligned} a &= \cos\phi/2 - i\,n_z\sin\phi/2 \\ b &= -(n_y + i n_x)\sin\phi/2 \end{aligned}$$

Note!

- $\mathrm{Det}\,\mathcal{D} = |a|^2 + |b|^2 = \cos^2\phi/2 + \sin^2\phi/2\,(n_x^2 + n_y^2 + n_z^2) = 1$

$\Rightarrow a = x + iy,\ b = z + iw \quad x^2 + y^2 + z^2 + w^2 = 1$

$\Rightarrow$ Space of spinor rotations

$\equiv$ Unit sphere in 4D $[S^3]$

- $$\mathcal{D}^\dagger \mathcal{D} = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} \overbrace{a^*a + b^*b}^{1} & \overbrace{\cancel{ba^* - a^*b}}^{0} \\ \underbrace{\cancel{ab^* - ab^*}}_{0} & \underbrace{b^*b + a^*a}_{1} \end{pmatrix}$$

$$= \mathbb{I}$$

$\mathcal{D}$ is unitary.

[Always, if A is Hermitian, e^{iA} is unitary

$$(e^{iA})^\dagger = e^{-iA^\dagger} = e^{-iA} = (e^{iA})^{-1}\ ]$$

Spinor Rotations are unitary, unimodular matrices

$\mathcal{D} \in SU(2)$.

Are all $SU(2)$ transformations spinor rotations?

$\mathrm{Dim}(\mathcal{D}) = 3$ [unit sphere in 4D, or $\hat{n} = 2$ dim $+ \varphi = 1$ dim]

$\mathrm{Dim}(SU(2)) = [4$ complex #s $\times 2$ (Re, Im) $-$ 4 constraints $U(2)$ $-$ 1 det $=1$ $] = 3$ ✓

Yes ✓ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overbrace{a^*a + c^*c}^{1} & \overbrace{ba^* + dc^*}^{2} \\ ab^* + cd^* & \overbrace{c^*c + d^*d}^{1} \end{pmatrix}$

$\boxed{\text{SO(3) and vector rotations}}$

Aren't rotations 3×3 real matrices? $SO(3)$?
What's this weird 2×2 complex matrix stuff?
$SU(2)$?

- Rotations are a group. [Identity, inverse, closed, associative]
  $g_1 * g_2 = g_3$

  Rotations act on vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$

  $\rightarrow$ 3×3 matrix representation of group.

$g \rightarrow R(g)$ s.t. $R(g_1)R(g_2) = R(g_3)$

- Rotations act on spinors $\begin{pmatrix}\alpha_\uparrow \\ \alpha_\downarrow\end{pmatrix} \in \mathbb{C}^2$ (spin ½ wavefcts)

  $\rightarrow$ 2×2 complex matrix representation

$g \rightarrow \mathcal{D}^{\frac{1}{2}}(g)$ $\quad \mathcal{D}^{\frac{1}{2}}(g_1)\mathcal{D}^{\frac{1}{2}}(g_2) = \mathcal{D}^{\frac{1}{2}}(g_3)$

- Rotations act on $\ell = 1$ spatial wavefunctions $\begin{pmatrix}\beta_{+1} \\ \beta_0 \\ \beta_{-1}\end{pmatrix} \in \mathbb{C}^3$

  $\beta_{+1} Y_1^1(\theta,\varphi) + \beta_0 Y_1^0(\theta,\varphi) + \beta_{-1} Y_1^{-1}(\theta,\varphi)$

$g \rightarrow \mathcal{D}^{(1)}(g)$ $\quad \rightarrow 3\times3$ complex rep

~~vectors~~ Note: p-orbitals $\quad p_z = \begin{pmatrix}0\\1\\0\end{pmatrix} = Y_1^0(\theta,\varphi) \sim \cos\theta$

$Y_1^1 \sim \sin\theta\, e^{i\varphi}$ $\quad Y_1^{-1} \sim \sin\theta\, e^{-i\varphi}$

$p_x = \sim \dfrac{Y_1^1 + Y_1^{-1}}{\sqrt{2}} = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

$p_y \sim \sim \dfrac{Y_1^1 - Y_1^{-1}}{\sqrt{2}i} = \begin{pmatrix}-i/\sqrt{2}\\0\\i/\sqrt{2}\end{pmatrix}$

Change basis Y_ℓ^m to $p_x, p_y, p_z \rightarrow \mathcal{D}^1(g) \rightarrow R(g)$ ∴ equivalent reps.

But then is $SU(2) = SO(3)$ as a group?

- Two groups G_1, G_2 are *isomorphic* if one-to-one map $g_1(g_2)$ preserves multiplication rule.

$\mathcal{D}(R)$ preserves multiplication rule.

Q: Is $\mathcal{D}(R)$ 1-1?

A: Actually, $\mathcal{D}(R)$ isn't well defined!
Two $SU(2)$ matrices for each $SO(3)$ matrix! 360° rotation $= \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$

Exercise 4.1

$R(\mathcal{D})$ is well-defined!
unit sphere S^3 wraps around $SO(3)$ twice.