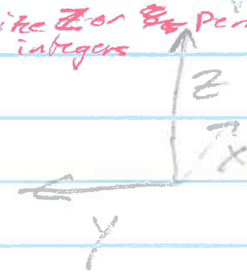
Lie Algebras and Angular Momentum

$SU(2)$, $SO(3)$ are Lie groups (continuous groups)
(as opposed to ~~permut~~ discrete groups like $\mathbb{Z}$ or $\mathbb{Z}_5$ permutation group S_5)
integers

Rotations don't commute

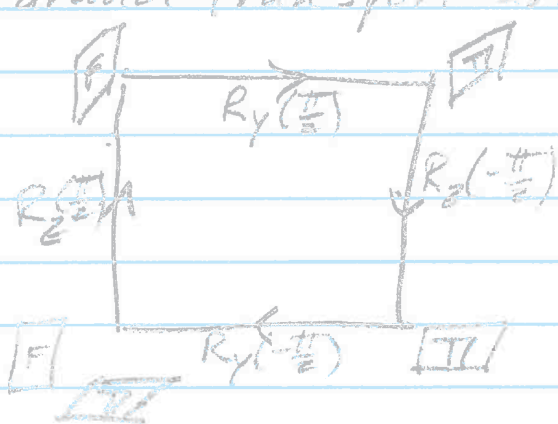


- Raise left hands, looking at palm
- Rotate +90° about $\hat{z}$
  (Right-hand rule:
  thumb along $\hat{z}$, positive φ sweeps fingers from flat to closed)

- Rotate +90 about $\hat{y}$
- Rotate −90 about $\hat{z}$
- Rotate −90 about $\hat{y}$

$$R^{[y]}(-\tfrac{\pi}{2})R^{[z]}(-\tfrac{\pi}{2})R^{[y]}(\tfrac{\pi}{2})R^{[z]}(\pi/2) \neq \mathbb{I}$$

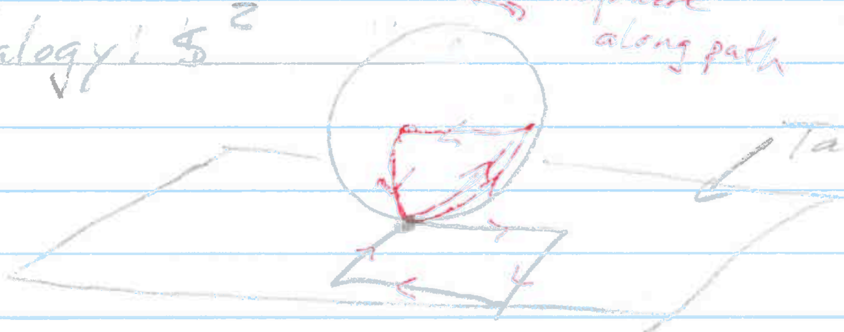
"Parallel transport"* around closed loop doesn't close in rotation space



* Not a precise analogy: fiber bundles
curvature ⟺ orbit in base space
commutator ⟺ orbit in fiber

What is the space with the square path?

Analogy: S^2



Tangent space to Lie group = <u>Lie Algebra</u>

- Linear vector space
- "Commutator" product (below)
- Basis = Infinitesimal symmetries

SO(3): Basis J_x, J_y, J_z

WARNING: Usual QM uses basis $m_z = {+1 \atop 0}_{-1}$, not xyz

<u>Vector version</u> $R_z(\underset{\text{small}}{\Delta}) = \begin{pmatrix} \cos\Delta & -\sin\Delta & 0 \\ \sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$J^{(z)} = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \approx \mathbb{1} + \Delta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\mathbb{1} - \frac{i\Delta J^{(z)}}{\hbar}}$$

analogy w/ $D = \mathbb{1} e^{-\frac{i S_z \phi}{\hbar}}$

$$J^{(x)} = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J^{(y)} = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\boxed{J^{(i)}_{jk} = -i\hbar\, \varepsilon_{ijk}}$$

$$\underset{\mathbb{1}}{e}^{-\frac{iJ^{(z)}}{\hbar}\Delta}$$

$$R^{[z]}_{jk}(\Delta) = \delta_{jk} - \frac{i J^{[z]}_{jk}}{\hbar}\Delta - \frac{J^{[z]}_{j\ell} J^{[z]}_{\ell k}}{2\hbar^2}\Delta^2$$

$+ i \ldots$

$$= \delta_{jk} - \varepsilon_{3jk}\Delta - \frac{\Delta^2}{2}\varepsilon_{3j\ell}\varepsilon_{3\ell k}$$

How can we quantify the non-commutativity?

- Curved surface?

  Curvature tensor = Parallel transport around tiny closed loop

- Lie group?

$$R^{[m]}(-\Delta)\, R^{[n]}(-\Delta)\, R^{[m]}(\Delta)\, R^{[n]}(\Delta)$$

$$= \left(\mathbb{1} + i\Delta \frac{J^{[m]}}{\hbar} - \frac{\Delta^2}{2\hbar^2} J^{[m]2}\right)\left(\mathbb{1} + i\Delta\frac{J^{[n]}}{\hbar} - \frac{\Delta^2}{2\hbar^2} J^{[n]2}\right)\left(\mathbb{1} - i\Delta \frac{J^{[m]}}{\hbar} - \frac{\Delta^2}{2\hbar^2} J^{[m]2}\right)\left(\mathbb{1} - i\Delta \frac{J^{[n]}}{\hbar} - \frac{\Delta^2}{2\hbar^2} J^{[n]2}\right)$$

$$= \mathbb{1} + \Delta(0) + \frac{\Delta^2}{\hbar^2}\Big(-J^m J^n + J^{m2} - J^m J^n + J^n J^m + J^{n2} + J^n J^m - \frac{J^{m2}}{2} - \frac{J^{n2}}{2} - \frac{J^{m2}}{2} - \frac{J^{n2}}{2}\Big)$$

$$= \mathbb{1} - \frac{2\Delta^2}{\hbar^2}\left[J^n, J^m\right]$$

$$[J^x, J^y] = (i\hbar)^2\left[\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} - \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\right]$$

$$= (i\hbar)^2\left[\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right] = i\hbar\left(i\hbar\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right)$$

$$= i\hbar J_z$$

In general

$$\boxed{[J^{[i]}, J^{[j]}] = i\hbar\, \epsilon_{ijk} J^{[k]}}$$

Same as for $[S_i, S_j] = i\hbar\, \epsilon_{ijk} S_k$

ϵ_{ijk} totally antisymmetric tensor

Note!

- Matrix forms for J^x, J^y, J^z different in $m_z = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ basis: $J_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

- Commutation relations depend only on group — must be same in m_z basis.

- Commutation relations for $SU(2)$

$$S^{[k]} = \frac{\hbar}{2} \sigma_k$$

$$[S^{[x]}, S^{[y]}] = \left(\frac{\hbar}{2}\right)^2 \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar}{2} \left(\frac{\hbar}{2} \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\} \right)$$

$$= i\hbar \left(\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = i\hbar S^{[z]}$$

$[S^{[i]}, S^{[j]}] = i\hbar \varepsilon_{ijk} S^{[k]}$ also.

(Makes sense: $SU(2) \simeq SO(3)$ near $\mathbb{I}$; double covering past $180^\circ \Rightarrow$ exercise 4.1).