'Second quantization' - commutation etc.

Last lecture:
* Fock space = occupation number representation
* Annihilation operator ai, reduces number in state i
* Creation operator increases number
* Multiply by sqrt(Ni) makes single-particle operators simple

$$\langle N_k - 1 | a_k | N_k \rangle = \sqrt{N_k} \qquad \langle N_i | a_i^+ | N_i - 1 \rangle = \sqrt{N_i}$$

$$V = \sum_i V(x_i) = \sum_{ik} V_{ik}\, a_i^+ a_k \qquad V_{ik} = \int \psi_i^*(x)\, V(x) \underbrace{\psi_k(x)}_{\text{Basis for number counting}} dk$$

$$U = \sum_{i<j} U(x_i, x_j) = \sum_{iklm} U_{iklm}\, a_i^+ a_k^+ a_l a_m \quad \ldots$$

Pick basis of position states.
* |0> = state with no particles

Note: $a_x^+, a_{x'}$ often written $\hat{\psi}^+(x), \hat{\psi}(x')$

- $|x\rangle = a_x^+ |0\rangle \Rightarrow |\psi\rangle = \int \psi(x)\, a_x^+ |0\rangle$
- $|\psi(x_1, x_2)\rangle = \int dx_1\, dx_2\, \psi(x_1, x_2)\, a_{x_1}^+ a_{x_2}^+ |0\rangle$

Bosons $\psi(x_1, x_2) = \psi(x_2, x_1) \Rightarrow a_{x_1}^+ a_{x_2}^+ = a_{x_2}^+ a_{x_1}^+ \Rightarrow [a_i^+, a_j^+] = 0$

Fermions $\psi(x_1, x_2) = -\psi(x_2, x_1) \Rightarrow a_{x_1}^+ a_{x_2}^+ = -a_{x_2}^+ a_{x_1}^+$

$$\Rightarrow \underbrace{\{a_i^+, a_j^+\}}_{\text{Anticommutator}} = a_i^+ a_j^+ + a_j^+ a_i^+ = 0$$

Fermions a_i^+, a_j usually called c_i^+, c_j

Hermitian Conjugate $\Rightarrow [a_i, a_j] = \{c_i, c_j\} = 0$

Q: Does $a_i^+ a_i$ change N_i? What is it?

A: $a_i^+ a_i |N_i\rangle = a_i^+ (\sqrt{N_i}\, |N_i - 1\rangle) = \sqrt{N_i}\sqrt{N_i}\, |N_i\rangle$

$a_i^+ a_i = N_i$ <u>Number Operator</u>

Remember, non-interacting particles energy ε_i

$$\mathcal{H} = \sum \varepsilon_i a_i^+ a_i = \sum_i \varepsilon_i N_i \quad \text{(makes sense)}$$

Q: What is $a_i a_i^+$? What is $[a_i, a_i^+]$?

A: $a_i a_i^+ \underbrace{|N_i\rangle}_{\substack{N_k - 1 \\ \text{in formula}}} = a_i \underbrace{\sqrt{N_i+1}}_{N_k}\, |N_i\rangle = (N_i+1)\,|N_i\rangle$

$$[a_i, a_i^+] = a_i a_i^+ - a_i^+ a_i = (N_i+1) - N_i = 1$$

$$\left.\begin{array}{ll} [a_i, a_j^+] = \delta_{ij}, & \{c_i, c_j^+\} = \delta_{ij} \\ [a_i, a_j] = 0 & \{c_i, c_j\} = 0 \\ [a_i^+, a_j^+] = 0 & \{c_i^+, c_j^+\} = 0 \end{array}\right\} \text{Commutation Relations}$$

* Hamiltonians almost always in second quantized form
* Gregarious bosons exercise:

$$a_r^+ = \Psi^+(\vec r) = \sum \underbrace{\langle \vec r | \psi_i \rangle}_{\substack{1/\sqrt{M} \\ M \text{ lowest states}}} a_i^+$$

$$a_r^+ (a_0^+)^{N_0} |0\rangle = \sum_{i=0}^{M-1} \frac{1}{\sqrt{M}}\, a_i^+\, |N_0\rangle$$

$$= \frac{1}{\sqrt{M}} \Big\{ \underbrace{\sqrt{N_0+1}}_{\substack{\text{Probability up by} \\ N_0+1\,!}}\, |N_0+1\rangle + \sum_{i=1}^{M-1} |N_0, 0, 0, \dots, N_i = 1, 0, \dots\rangle \Big\}$$

Bosons like to collect in same state:

Lasers, superfluids

[View exercise as new derivation $a_i^+ |N_i\rangle = \sqrt{N_i+1}\, |N_i\rangle$]

* Off-diagonal long-range order extra credit exercise:
  apply a, adag to ground state of many-particle soup

$\langle \Psi | a^{+}(r') a(r) | \Psi \rangle$ = 'reduced' density matrix (tracing over all but one particle)