Time Dependent Perturbations and Fermi's Golden Rule

Time-dependent perturbation on time-independent Hamiltonian

$$\mathcal{H}(t) = \mathcal{H}_0 + V(t)$$

* AC (classical) electromagnetic fields: absorption and stimulated emission
* Slowly varying fields: the adiabatic theorem and Berry's phase
* Rapidly varying fields: the sudden approximation

* Turning on interactions and watching the decay of excited states
  - Excited atom + coupling to electromagnetic waves = decay rate via photon emission
  - Uranium nucleus + transmission through Coulomb barrier = alpha-decay rate
  - Electron on quantum dot + hopping to lead = metastable state
  - Electron-hole excitation of noninteracting electron gas + e-e interactions = Quasiparticle lifetimes

[No actual time dependence! Need to start without interactions: will use formal trick

V(t) = lim_eta->0 V exp(eta t)]

Perturb in V

$$\mathcal{H} = \mathcal{H}_0 + \underbrace{\varepsilon}_{} V(t)$$

$V(t)$ general fct x, p, t

$\mathcal{H}_0 |n\rangle = E_n |n\rangle$

Work in unperturbed basis of $\mathcal{H}_0$

General initial state

$$|\alpha\rangle = \sum c_n(0) |n\rangle$$

Boring $\mathcal{H}_0$ dependence

$$\mathcal{U}(t)|\alpha\rangle = \sum \underbrace{c_n(t)}_{} \overbrace{e^{-iE_n t/\hbar}}^{} |n\rangle$$

New Occupation Amplitudes

Transitions.

Expand $c_n(t)$ in formal power series in ε

Q: What is $c_n^{(0)}(t)$, $c_n(t)$ for $\varepsilon = 0$?

A: Since $\mathcal{U}_0(t)|n\rangle = e^{-i\mathcal{H}t/\hbar}|n\rangle$, $c_n^{(0)}$ is constant

$$c_n(t) = c_n^{(0)} + \varepsilon c_n^{(1)}(t) + \varepsilon^2 c_n^{(2)}(t)$$

Schrödinger's Eqn $\quad i\hbar \frac{d|\alpha\rangle}{dt} = (\mathcal{H}_0 + V(t))|\alpha\rangle$

$\rightarrow$ Equations for $c_n^{(1)}, c_n^{(2)}, \ldots$

Fixed in time

$$\frac{d|\alpha\rangle}{dt} = \sum_n \left(\frac{dc_n^{(0)}}{dt} + \varepsilon \frac{dc_n^{(1)}}{dt} + \varepsilon^2 \frac{dc_n^{(2)}}{dt}\right) e^{-iE_n t/\hbar} \overbrace{|n\rangle}^{}$$

$$\cancel{+ (c_n^{(0)} + \varepsilon c_n^{(1)} + \varepsilon^2 c_n^{(2)})\left(\frac{-iE_n}{\hbar}\right) e^{-iE_n t/\hbar} |n\rangle}$$

$$= \frac{1}{i\hbar}(\mathcal{H}_0 + \varepsilon V)|\alpha(t)\rangle$$

$$= \sum_i (c_i^{(0)} + \varepsilon c_i^{(1)} + \varepsilon^2 c_i^{(2)}) \left(-\frac{i}{\hbar}\right)(\overset{E_n}{\cancel{\mathcal{H}_0}} + \varepsilon V) e^{-iE_i t/\hbar} |i\rangle$$

Q: What equation does $\varepsilon = 0$ give?

A: $\frac{dc_n^{(0)}}{dt} e^{-iE_n t/\hbar} |n\rangle = 0 \quad \Rightarrow \frac{dc_n^{(0)}}{dt} = 0$

Take both sides times $\langle n| e^{iE_nt/\hbar}$:

$$\varepsilon \frac{dc_n^{(1)}}{dt} + \varepsilon^2 \frac{dc_n^{(2)}}{dt} = -\frac{i}{\hbar} \sum_i \left[\varepsilon c_i^{(0)} + \varepsilon^2 c_i^{(1)}(t)\right] \underbrace{\langle n| e^{+iE_nt/\hbar} V e^{-iE_it/\hbar} |i\rangle}_{V_{ni}\, e^{i(E_n - E_i)t/\hbar} \; = V_{ni}\, e^{i\omega_{ni}t}}$$

$\omega_{ni} = (E_n - E_i)/\hbar$ = Frequency difference

$$e^{i\mathcal{H}_0 t/\hbar}\, V\, e^{-i\mathcal{H}_0 t/\hbar} = V_I$$: Interaction Picture

(Like Heisenburg representation
Useful in field theory)

Q: What is $\frac{dc^{(1)}}{dt}$?

A: $\frac{dc^{(1)}}{dt} = \left(-\frac{i}{\hbar}\right) \sum_n V_{ni}\, e^{i\omega_{ni}t}\, c_i^{(0)}$

Q: If at $t=0$, $|\alpha\rangle = |i\rangle$ (so $c_n^{(0)} = \delta_{ni}$), what is $c_1(t)$?

A: $c^{(1)}(t) = -\frac{i}{\hbar} \int^t e^{i\omega_{ni}t'} V_{ni}(t')\, dt'$

Now consider $V(t) = e^{\eta t} V$, gradually turning on potential, as $\eta \to 0$. For $n \neq i$,

$$c_n^{(1)}(t) = -\frac{i}{\hbar} V_{ni} \int^t e^{\eta t'} e^{i\omega_{ni} t'} dt' = -\frac{i}{\hbar} V_{ni} \frac{e^{\eta t + i\omega_{ni} t}}{\eta + i\omega_{ni}}$$

$c_n^{(1)}(t)$ is amplitude for decay into state n after time t

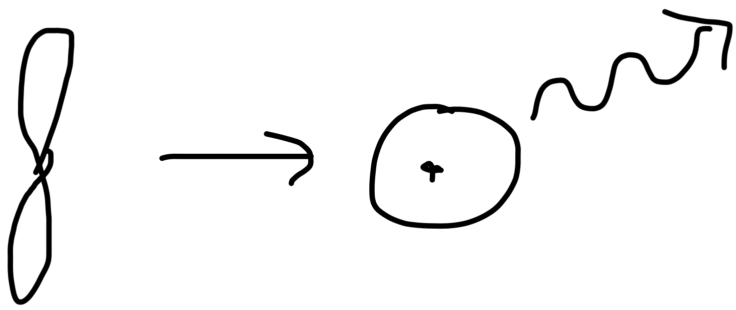
Q: What is the decay rate, $\frac{d|c_n^{(1)}(t)|^2}{dt}$?

A: $|c_n(t)|^2 = \frac{|V_{ni}|^2}{\hbar^2} \frac{e^{2\eta t}}{\eta^2 + \omega_{ni}^2} \Rightarrow \frac{d|c_n(t)|^2}{dt} = \frac{2|V_{ni}|^2}{\hbar^2} \frac{\eta}{\eta^2 + \omega_{ni}^2} e^{2\eta t}$

But as $\eta \to 0$, this vanishes unless $\omega_{ni} = 0$?
(Energy conservation. Adiabatic theorem.)

If $\omega_{ni} = 0$, degenerate perturbation ...

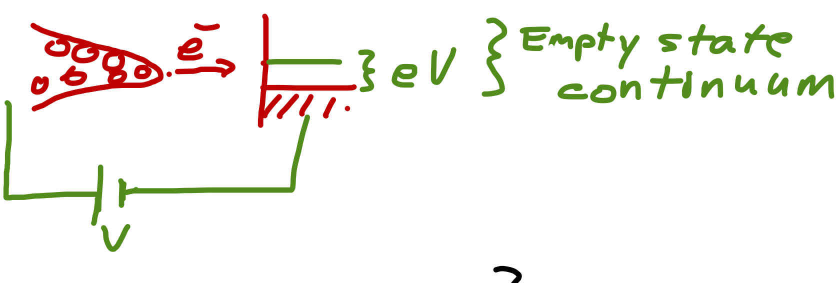
But, what if there are a continuum of final states?



Continuum of final photon energies

H 2p state $\to$ 1s + photon

STM e^- tunnel into metal



$$\frac{d|c_n(t)|^2}{dt} = \lim_{\eta \to 0} \frac{|V_{ni}|^2}{\hbar^2} \frac{2\eta}{\eta^2 + \omega_{ni}^2} e^{2\eta t}$$

Q: What is $\lim_{\eta\to 0} \frac{\eta}{\eta^2+\omega^2}$? Where have you seen this?

A: Exercise 2.2; sharply peaked, integral = $\arctan\Big|_{-\infty}^{\infty} = \pi$,

$$\lim_{\eta\to 0} \frac{\eta}{\eta^2+\omega^2} = \pi\delta(\omega)$$

$\delta(\omega_{ni}) = \hbar\,\delta(E_n - E_i)$, so

$$\frac{d|c_n(t)|^2}{dt} \simeq \frac{2\pi}{\hbar}|V_{ni}|^2\,\delta(E_n - E_i) \qquad \text{Fermi's Golden Rule}$$

Total decay rate, sum (integrate) over final states.

If V_{ni} varies smoothly over energy $V_{ni} = V_i(E_n)$

$$\frac{dP}{dt} = \frac{2\pi}{\hbar}|V_i(E_i)|^2 \underbrace{\int dn\, \delta(E_n - E_i)}_{\text{Density of Final States at } E_i}$$