Gauge Invariance, Phase, and Charge Conservation

Emmy Noether's Theorem:

Symmetries <-> Conservation Laws

(usually derived in classical mechanics using Lagrangians)

Time Translations <-> Energy

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle \quad \Rightarrow$$

$$|\psi(t+dt)\rangle \doteq \left(\mathbb{1} - \frac{iH}{\hbar}dt\right)|\psi(t)\rangle$$

o

$$|\psi(t+\tau)\rangle = e^{-iH\tau/\hbar}\,|\psi(t)\rangle$$

Spatial translations <-> Momentum

$$\underbrace{|\psi(x-dx)\rangle}_{\text{move }\psi\text{ right by }dx} = \left(\mathbb{1} - dx\frac{\partial}{\partial x}\right)|\psi(x)\rangle = \left(\mathbb{1} - \underbrace{\frac{ip}{\hbar}}_{}dx\right)|\psi(x)\rangle$$

$$p = -i\hbar\frac{\partial}{\partial x}$$

$$|\psi(x+\Delta)\rangle = e^{-\frac{ip\Delta}{\hbar}}\,|\psi(x)\rangle$$

Rotations <-> Angular Momentum

Rotations generated by angular momentum J

(Infinitesimal rotation, Lie algebra, later)

$$\mathbb{1} - \frac{i}{\hbar}d\varphi\cdot J$$

$$R = e^{-\frac{i}{\hbar}J\cdot\vec{\varphi}}$$

$\vec{\varphi}$ rotate $|\psi\rangle$ about axis $\hat{\varphi}$

What about wavefunction phase?

$$\psi' = e^{-i/\hbar\,\chi(x)}\psi$$

- Promote global symmetry to local one
- Each position gets own reference 'zero' phase $-\chi(x)$

But Schrodinger involves gradients, connecting neighboring points

$$i\hbar \frac{\partial \psi'}{\partial t} = \cancel{e^{-i/\hbar\,\chi}}\left(i\hbar \frac{\partial \psi}{\partial t}\right) \stackrel{?}{=} \frac{\hbar^2}{2m} \underbrace{e^{i/\hbar\,\chi}\nabla^2\left(e^{-i/\hbar\,\chi}\right)}_{\neq \nabla^2 \psi}\psi + \cancel{e^{-i/\hbar\,\chi}}\, V\psi$$

Want new 'covariant' derivative D:

$$e^{i/\hbar\,\chi(x)} D\, e^{-i/\hbar\,\chi} = \nabla \Rightarrow D\psi' = e^{-i/\hbar\,\chi}\nabla e^{+i/\hbar\,\chi(x)}\psi'$$
$$= -\left(i/\hbar\, \nabla\chi\right)\psi' + \nabla\psi'$$

$$D = \left(\nabla + \frac{i}{\hbar}\nabla\chi\right)$$

But this is just how A-field from E&M couples (B=curl A):

$$\left(\nabla + \frac{iq}{\hbar c}\right)^2 \psi$$

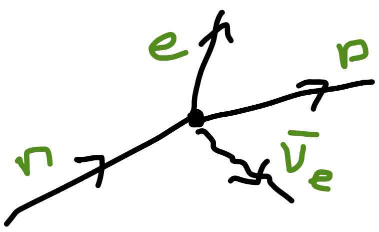
$q = -e$ for electron

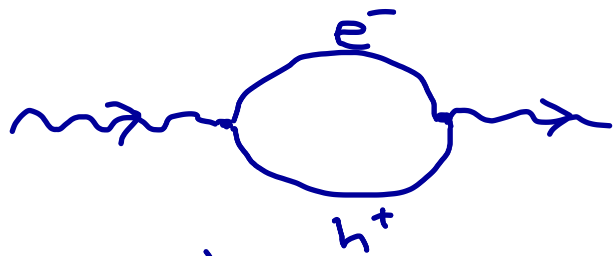
Call $\Lambda(x) = -\frac{q}{c}\chi(x)$

$$A' = A + \vec{\nabla}\Lambda$$

Gauge transformation <->
phase of wavefunction

But what's conserved? Deep connection to charge conservation (but need particle creation & annihilation)





Total Phase $\chi = -\frac{\Lambda(x)}{\hbar c}\sum q_i$

Charge Conservation
$\sum q_i$ same
Charge unchanged

Superfluid helium, Bose condensation, Lasers: Particles delocalize
Number of particles N(x) in a small volume indeterminate:
'particle conservation' law locally broken: broken gauge invariance!
Order parameter = Complex number

$$\psi(x) = \underbrace{|\psi|}_{\text{superfluid Density}} e^{i\chi}$$

Phase $= e^{iN\chi}$ of N-boson sector

Superconductors: Cooper pairs as bose 'particles' condensing.
Cooper pairs are charged?
Broken Gauge invariance gives "Goldstone" mode that 'eats'
photon, gives Meissner effect (expulsion of magnetic field).
The original Higgs mechanism!