50 Years of the Renormalization Group

Dedicated to the Memory of Michael E Fisher

Editors
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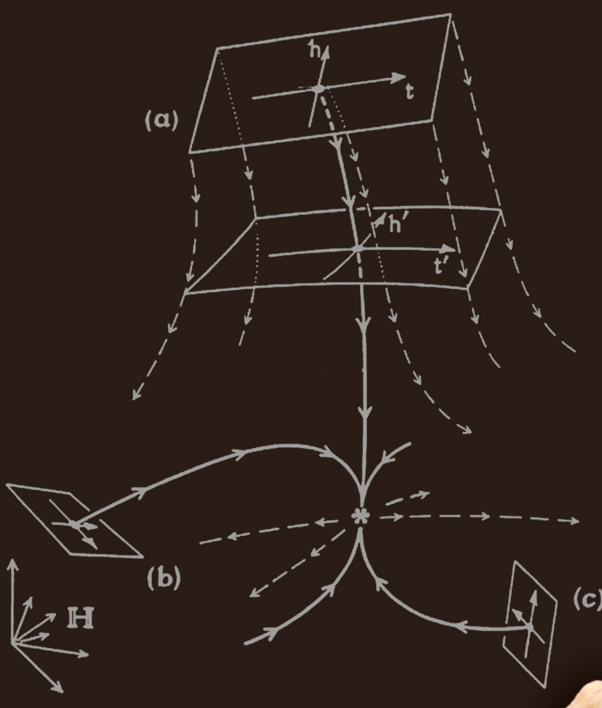

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Chapter 26

Normal Forms, Universal Scaling Functions, and Extending the Validity of the RG

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Our community has a deep and sophisticated understanding of phase transitions and their universal scaling functions. We outline and advocate an ambitious program to use this understanding as an anchor for describing the surrounding phases. We explain how to use normal-form theory to write universal scaling functions in systems where the renormalization-group flows cannot be linearized. We use the 2D Ising model to demonstrate how to calculate high-precision implementations of universal scaling functions and how to extend them into a complete description of the surrounding phases. We discuss prospects and challenges involved in extending these early successes to the many other systems where the RG has successfully described emergent scale invariance, making them invaluable tools for engineers, biologists, and social scientists studying complex systems.

1. Introduction

Half a century ago, Michael Fisher, together with Wilson and Kadanoff, introduced the renormalization group to analyze systems with emergent, fractal scale invariance. For five decades, physicists have applied these techniques to equilibrium phase transitions, avalanche models, glasses and disordered systems, the onset of chaos, plastic flow in crystals, surface morphologies, etc. But these tools have not made a substantial impact on engineering and biology. We believe it is our duty to make these tools accessible to the broader science community.

• We need to provide them with tools that allow them to describe not only the critical point but also properties of systems that exhibit incipient scale-invariant fluctuations yet are far from the critical region. These demand that we understand *corrections to scaling*, which become more important farther from the critical point. In Section 3, we build on Michael's work on analytic corrections to scaling with

Aharony[1,2] and his work on the complex analytic properties of the Onsager solution[3] to extend our understanding of the 2D Ising critical point to a full description of the ferromagnetic and paramagnetic phases.[4]

• We need not only universal power laws but also a complete description of all behavior of the material. These demand convenient access to accurate *universal scaling functions* that govern the behavior involving more than two quantities at a time. Michael taught us about these powerful tools in his pedagogical reviews and lectures. So, magnetization as a function of temperature goes as $M \sim t^\beta$, but magnetization as a function of temperature and external field goes as $M \sim t^\beta \mathcal{M}(h/t^{\beta\delta})$, where $\mathcal{M}(X)$ is a universal function, in principle predicted from the renormalization group. In Section 2, we discuss the correct way[5] of writing universal scaling functions for systems where logarithms and exponentials are found in addition to power laws (e.g., in the upper and lower critical dimensions). In Section 4, we present a userfriendly solution[6] for $\mathcal{M}(X)$ in the 2D Ising model, with a systematic expansion that captures the correct singular behaviors with seven-digit accuracy.

• We need rapidly converging methods that can connect our deep understanding of the singularities in phases and in universal scaling functions to quantitative predictions which make the best use of limited information away from the critical points. In modern numerical analysis, one can integrate or approximate analytic functions of one variable with exponential accuracy [7, Sections 4.6 and 5.8.1] so long as one knows the singularities at the end points. Chebyshev, Gauss, and Romberg methods have superseded Simpson's rule and its generalizations for approximating and integrating analytic functions and can be adapted to capture known singularities. We know that all properties are analytic in phases, and our mission for five decades has been to understand the singularities between phases. Can we explain the phases using the critical points? The exponential convergence in Sections 3 and 4 demonstrate that the normal-form theory can do so.

Focusing on the 2D Ising model allows us to show proof of principle, but can we aspire to similar progress for other less well-studied critical points? We suggest this as a key task for the next stage of research in critical phenomena and emergent scale invariance. In Section 5, we discuss progress by those using nonperturbative functional renormalization-group (NPFRG) methods,[8] which have found broad application not only in equilibrium thermodynamic systems but also in avalanche models, quantum systems, turbulence, etc. They explicitly coarse-grain and renormalize a system in fixed dimension, implicitly calculating the universal scaling functions and the system-specific behavior far from criticality. They have mostly been used to extract high-precision critical exponents, amplitude ratios, and proofs of success. If we can organize these calculations into user-friendly universal scaling functions, we may provide experimentalists, simulators, and theorists in a variety of fields with tools to describe matter beyond systems tuned to a single point on the phase diagram.

In Section 6, we embark into deep issues in the renormalization group and the prospects and challenges they provide to our mission to make the theory of critical phenomena an organizing principle for much of science.

2. Normal-Form Theory, RG Flows, and Logs

The renormalization group takes an enormous leap of abstraction — studying emergent scale invariance as a flow under coarse-graining in the space of all possible systems. This reduces the problem to the study of fixed points for differential equations in an infinite-dimensional space (e.g., free energy f, temperature $t = (T - Tc)$, field h, other parameters $u, \ldots$). Near the critical point for the Ising model, coarse-graining and rescaling the free energy f by a factor e^{ℓ} is described by the differential equations

$$\begin{aligned} df/d\ell &= df + At^2 + \text{other non-linear} \ldots, \\ dt/d\ell &= \lambda_t t + \text{non-linear} \ldots, \\ dh/d\ell &= \lambda_h h + \text{non-linear} \ldots, \\ du/d\ell &= \lambda_u u + \text{non-linear} \ldots \end{aligned} \tag{1}$$

The first step in most treatments of these RG flows is to linearize the flow near the fixed point. Positive eigenvalues λ_t and λ_h correspond to relevant operators like field h and temperature t; negative eigenvalues λ_u correspond to 'irrelevant' perturbations like u that provide *singular corrections to scaling* that are subdominant near the critical point. Using this linearization, this treatment then argues for universal power laws for things like the correlation length $\xi(t) \sim t^{\nu} = t^{1/\lambda_t}$ at $h = 0$, and universal power laws times universal functions

$$f(t, h, u) \sim t^{d\nu} \mathcal{F}(h/t^{\beta\delta}, ut^{\omega\nu}) = t^{d/\lambda_t} \mathcal{F}(h/t^{\lambda_h/\lambda_t}, ut^{-\lambda_u/\lambda_t}), \tag{2}$$

$$M(t, h, u) \sim t^{\beta} \mathcal{M}(h/t^{\beta\delta}, ut^{\omega\nu}) = t^{(d-\lambda_h)/\lambda_t} \mathcal{M}(h/t^{\lambda_h/\lambda_t}, ut^{-\lambda_u/\lambda_t}) \tag{3}$$

for things like the field-dependent free energy and magnetization, which describe the relations between more than two quantities.

This linearization is not useful in many cases (e.g., the 1D, 2D, and 4D Ising models and all models in their upper and lower critical dimensions). How to systematically formulate universal scaling functions in these cases has hitherto been mysterious. In this section, we describe the use of normal-form methods from dynamical systems theory by Raju *et al.*[5] to understand when this linearization is possible and how to modify the invariant arguments to the universal scaling functions functions when it is not.

Wegner and co-workers[9,10] in the early days justified this linearization by changing variables to *non-linear scaling fields* which transform linearly under the renormalization group. Cardy[11] denotes these new variables u_t and u_h, but we shall use tildes $\widetilde{t}$ and $\widetilde{h}$. By choosing a suitable Taylor expansion of the change of variables,

$$\begin{aligned} t(\widetilde{t}, \widetilde{h}, \widetilde{u}, \ldots) &= \widetilde{t} + a_{tu}\widetilde{t}\widetilde{u} + a_{th^2}\widetilde{t}\widetilde{h}^2 + \cdots, \\ h(\widetilde{t}, \widetilde{h}, \widetilde{u}, \ldots) &= \widetilde{h} + b_{hu}\widetilde{h}\widetilde{u} + b_{ht}\widetilde{h}\widetilde{t} + \cdots, \\ &\cdots \end{aligned} \tag{4}$$

the equations simplify to $d\widetilde{f}/d\ell = d\widetilde{f}$, $d\widetilde{t}/d\ell = \lambda_t \widetilde{t}$, $d\widetilde{u}/d\ell = \lambda_u \widetilde{u}$, etc. Aharony and Fisher[1,2] noted 10 years later that this change of variables leads to what we call *analytic corrections to scaling*, again subdominant near the critical point. The analytic corrections to scaling have power laws that involve integers and combinations of existing critical exponents β, ν, δ, and include an analytic background to the free energy; these corrections can be written in terms of derivatives of the universal scaling function. The aforementioned singular corrections to scaling introduce new critical exponents and become independent variables in the universal scaling functions.

Dynamical systems theory[5,12] tells us that linearizing the flow can only be done for what are called hyperbolic fixed points. The change of variables $f, t, h, u \to \widetilde{f}, \widetilde{t}, \widetilde{h}, \widetilde{u}$ is calculated one polynomial order at a time, and its radius of convergence can be a subtle mathematical issue.[13] In Section 3, we take on the ambitious task of attempting to describe the entire surrounding paramagnetic and ferromagnetic phases for the 2D Ising model. There, we shall see that convergence is tricky, but a good choice of coordinates can yield a radius of convergence that appears to converge precisely in the range from zero to infinite temperature.

Even for the two-dimensional Ising model, the RG fixed point cannot be linearized. The specific heat has a logarithmic singularity: often deemed $\alpha = 0(\log)$, but incompatible with a linearized flow. This arises because in 2D no polynomial change can remove the At^2 term in the flow of f in Eq. (1) (although rescaling the relative magnitudes of f and t can change the value of A to one). Somewhat messy algebra can confirm that this is due to a integer *resonance* between the two linear eigenvalues $\lambda_f = d = 2 = 2/\nu = 2 * \lambda_t$ for the 2D Ising model. The simplest normal form is thus $d\widetilde{f}/d\ell = 2\widetilde{f} - \widetilde{t}^2$, $d\widetilde{t}/d\ell = \widetilde{t}$, etc. This results in a singularity in the free energy of the form $t^2 \log(t^2)$ which will play an important role in Sections 3 and 4.

For the 4D Ising model, the leading irrelevant operator u becomes marginal, with a zero eigenvalue λ_u. There, it has always been clear that one cannot linearize the RG flow. In the dynamical systems nomenclature, the flow undergoes a transcritical bifurcation in $d = 4$ (together with the same resonance $d = 2/\nu$ seen in the 2D Ising above). Our analysis[5] shows that the normal form for the RG flows is[a]

$$d\widetilde{f}/d\ell = 4\widetilde{f} - \widetilde{t}^2, \quad d\widetilde{t}/d\ell = 2\widetilde{t} - A\widetilde{u}\widetilde{t}, \quad d\widetilde{h}/d\ell = 3\widetilde{h}, \quad d\widetilde{u}/d\ell = -\widetilde{u}^2 + D\widetilde{u}^3. \tag{5}$$

[a]The non-linear term proportional to $\widetilde{u}\widetilde{h}$ in the equation for $d\widetilde{h}/d\ell$ needed in the general normal form is known to be zero for the 4D Ising model. Raju[14] has shown this is because there is a redundant variable proportional to the magnetization cubed in the renormalization group (see Section 6).

The universal scaling of the free energy in a system of length L does not take the usual scaling form $f(t,h,u,L) = L^{-d}\mathcal{F}(X,Y,Z) + f_a(t,h,u,L)$ with $X = \widetilde{t}L^2$, $Y = \widetilde{h}L^3$, and $Z = \widetilde{u}/L^{\omega\nu}$. First, we have quite unusual scaling variables

$$X = \widetilde{t}L^2(W(yL^{1/D})/(1/(D\widetilde{u})-1))^{-A}, Y = \widetilde{h}L^3, \quad \text{and} \quad Z = yL^{1/D}, \tag{6}$$

Second, the free energy has a more complex form

$$f(t,h,u,L) = L^{-d}\mathcal{F}(X,Y,Z) - W(Z)^{-A}\left(\frac{W(Z)^{-A}}{1-A} - \frac{1}{A}\right) + f_a(t,h,u,L). \tag{7}$$

where $W(x)$ is the Lambert W or product log function, $\widetilde{u}$ is the marginal quartic term in the Landau free energy, y is a messy known function of $\widetilde{u}$, A and D are the amplitudes of non-linear terms in the renormalization-group flow (Eq. (5)), and f_a is a non-singular, analytic function near the critical point.

Equations (6) and (7) capture the complete, correct singularity for the 4D Ising model; the traditional log and log–log corrections in the specific heat and the susceptibility arise from expansions of the Lambert W function for large arguments. At bifurcations like $d = 4$ and resonances as at $d = 2$, normal-form theory dictates the nature of the singularity at the critical point.

It *also* tells us that the free energy in the surrounding phase can be found by changing variables. So, for example, we know that the liquid–gas critical point at high pressures and temperatures is in the 3D Ising universality class. Hence, the liquid and gas phase properties should be given by

$$f(T,P) = \widetilde{t}(T,P)^{3\nu}\mathcal{F}_{3\text{DIsing}}(\widetilde{h}(T,P)/\widetilde{t}(T,P)^{\beta\delta}, \widetilde{u}(T,P)t^{\omega\nu}, \dots) + f_a(T,P), \tag{8}$$

where f_a, $\widetilde{t}$, $\widetilde{h}$, and $\widetilde{u}$ are analytic in temperature and pressure near the critical point. As usual, from the free energy (and a corresponding scaling form for the correlation function), one can derive all of the equilibrium and linear-response properties of the resulting phases. We shall implement a change of coordinates like this for the 2D Ising model in Section 3.

We highlight the use of our normal-form methods[5] to solve a 20-year-old puzzle — the unusual scaling of avalanches in the non-equilibrium 2D random-field Ising model[15] (see Fig. 1). Over 25 years ago, we used the random-field Ising model to study avalanches. We could understand the scaling in 3, 4, 5, 7, and 9 dimensions,[16] but two dimensions made no sense. Changing the arguments from ratios of power laws to the invariant functions determined from normal-form theory was one part of the puzzle.[b]

The distribution of avalanche sizes in the random-field Ising model is cut off at size $\Sigma(w)$ depending on the disorder w. In three dimensions and higher, it takes the traditional form $\Sigma(w) = w^{-d_f\nu}$. In two dimensions, normal-form theory predicts that $\Sigma = (B + 1/w)^{-Bd_f+C}\exp(d_f/w)$, with d_f a universal critical exponent, and B and C being universal constants associated with irremovable nonlinear terms

[b]Using random lattices to suppress faceting was the other obstacle.

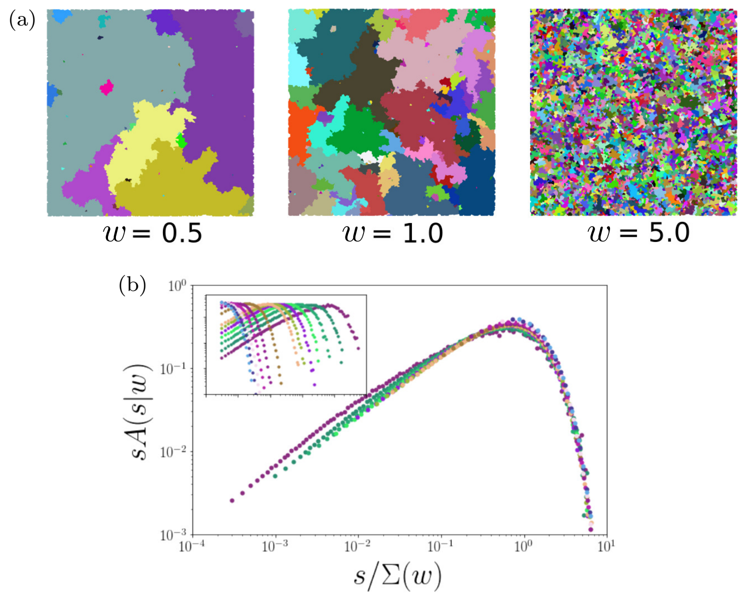

Fig. 1. Avalanche sizes for the 2D $T = 0$ random-field Ising model (from Ref. 15). (a) Avalanches for disorders $w = 0.5$, 1.0, and 5.0; each color is a separate avalanche. (b) Scaling collapse of the area-weighted avalanche size distribution. Note the factor of 10 range in disorder w, and the factor of 2000 range in typical avalanche size Σ. Here, the invariant scaling combination $\Sigma(w) = \Sigma_s(B + 1/w)^{-Bd_f+C}\exp(d_f/w)$ is not a ratio of power laws, but is derived directly from the non-linear normal form of the renormalization-group equations in the lower critical dimension (as in Eq. (5)). The area-weighted avalanche size distribution is thus $sA(s|w) = (s/\Sigma(w))^x\mathcal{A}((s/\Sigma(w))^y)$, with $\mathcal{A}$ a universal scaling function.

in the renormalization-group flow. As shown in Fig. 1, this does an excellent job explaining the behavior.

3. Changing Coordinates to Describe Phases: Matching Onsager

As foreshadowed in Section 2, normal-form theory tells us how to transform universal scaling forms to describe the entire phases surrounding a critical point. This section summarizes our recent results,[4] which implement this for the 2D Ising model in zero field, where Onsager's exact solution[17] for the free energy enables quantitative validation of the results.

Given the asymptotic scaling form for the free energy (or any other quantity) in normal-form coordinates, $\widetilde{f}(\widetilde{t}, \widetilde{h}, \widetilde{u})$, the free energy as a function of physical temperature, field, and irrelevant perturbations is $f(t, h, u) = \widetilde{f}(\widetilde{t}(t, h, u), \widetilde{h}(t, h, u), \widetilde{u}(t, h, u)) + f_a(t, h, u)$. The functions $\widetilde{t}(t, h, u)$ and $\widetilde{h}(t, h, u)$ are given by precisely the analytic change of variables that inverts Eq. (4), mapping the non-linear scaling variables back to their physical counterparts. Additionally, we must also add terms $f_a(t, h, u)$ accounting for the analytic background of the free energy. Note that the change of coordinates are linear at lowest order, e.g., $\widetilde{t} \sim t$ and $\widetilde{h} \sim h$. Therefore, $f(t, h, u) \sim \widetilde{f}(t, h, u)$ near the critical point; in other words, the normal-form free energy $\widetilde{f}$ is the asymptotic scaling form near the critical point, usually computed using RG and related techniques.

We have already seen that the 2D Ising model in zero field has a non-linear normal form due to the resonance between the free energy and temperature, $d\widetilde{f}/d\ell = 2\widetilde{f} - \widetilde{t}^2$, $d\widetilde{t}/d\ell = \widetilde{t}$, and that these flow equations give rise to a logarithmic singularity, $\widetilde{f} = -\widetilde{t}^2 \log(\widetilde{t}^2)$. Following the procedure outlined above, the free energy as a function of temperature is simply

$$f(t) = -\widetilde{t}(t)^2 \log(\widetilde{t}(t)^2) + f_a(t). \tag{9}$$

This result is consistent with Onsager's exact solution, which is known to have the form $a(t) \log t^2 + b(t)$ for some analytic functions a and b. Comparing these expressions for the free energy, we need to find the coordinate change $\widetilde{t}(t)$ and analytic background $f_a(t)$ by solving

$$a(t) = -\widetilde{t}(t)^2, \quad b(t) = \widetilde{t}(t)^2 \log(\widetilde{t}(t)^2/t^2) + f_a(t). \tag{10}$$

Note that because $\widetilde{t}(t)$ is linear to lowest order in t, the term $\log\left(\widetilde{t}(t)^2/t^2\right)$ is indeed analytic.

We recently[4] computed the free energy, Eq. (9), by perturbatively expanding Onsager's exact free energy around the critical point. A key question is the radius of convergence of the coordinate transformation $\widetilde{t}(t)$ and $f_a(t)$ to the normal form. Unlike Taylor expansions about analytic points, the radius of convergence of this normal-form analytic expansion about a singular point is not simply the distance in the complex plane to the next-nearest singularity. One might hope that physics would govern the convergence — perhaps the distance to zero temperature, infinite temperature, or the nearest other phase transition. Indeed, in each of the expansions discussed in the following, the critical point is closest to zero temperature, with this distance determining the radius of convergence.

Our investigations showed the importance of the choice of variable used to parameterize the distance to the critical point (Fig. 2(a)). Expanding $\widetilde{t}$ and f_a in temperature $t = (T - T_c)$, for example, converges in an estimated range $-T_c < t < T_c$ (or $0 < T < 2T_c$). This is not the only natural choice, however. The low-temperature expansions for the Ising model are expressed in powers of $X = \exp(-2/T)$. X is also a natural variable in that the zeros of the 2D Ising partition function in the complex X plane form a circle passing through X_c, as Fisher explained.[3] Using the Onsager solution to expand $\widetilde{t}$ and f_a in terms of $x = X - X_c$ yields a radius of convergence that extends all the way from zero temperature to $X = 2X_c$ (corresponding to $T \approx 4.7\, T_c$), but fails to describe higher temperatures.

Here, we used special properties of the 2D Ising model to identify a new variable

$$V = \frac{5 - 3\sqrt{2} + X}{1 + \sqrt{2} + X} \quad v = V - V_c, \tag{11}$$

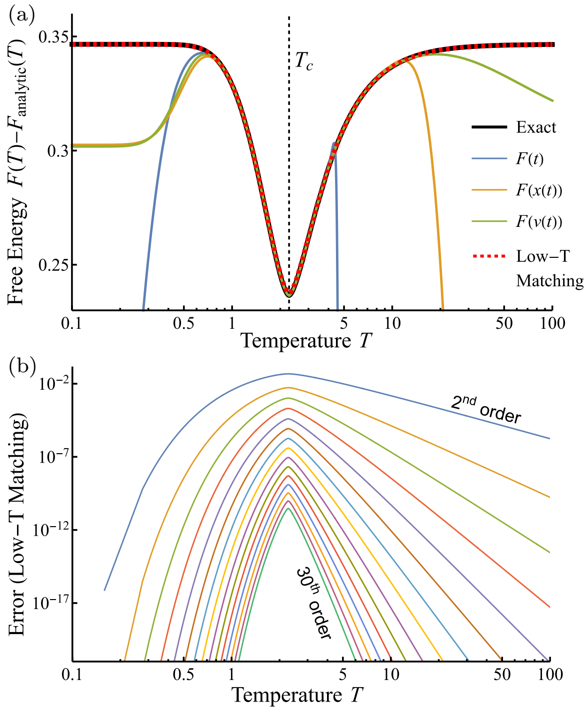

Fig. 2. Capturing the entire phase with analytic corrections, 2D Ising (from Ref. 4). (a) Radius of convergence depends on the expansion variable: Expanding Onsager's exact free energy (black line) in t, x, and v (colored solid lines) leads to increasing radii of convergence, with the convergence of the v-expansion estimated to cover the entire physical temperature range. Each expansion shown is 20th order. Determining analytic corrections in v by matching to the low-temperature expansion does even better, accurately reproducing the free energy across all temperatures even at low orders (red dashed line, sixth order). (b) Exponential convergence: Adding analytic corrections to the universal scaling function at T_c by fitting to the low-temperature expansion results in exponential convergence to the Onsager solution as we add more terms.

which allows our estimated radius of convergence to cover the full physical temperature range from zero to infinity (Fig. 2(a)). Our coordinate v, unlike t or x, respects the self-dual symmetry of the 2D Ising model. Fisher's circle of zeros in X (and thus x) breaks this self-dual symmetry. The linear fractional transformation in Eq. (11) precisely unwraps this circle of zeros into a straight line, extending the self-dual symmetry to the complex plane. The circle of zeros becomes the branch cut of the logarithm in the scaling function for the free energy (Eq. (9)).

Figure 2(a) compares the expansions $f(t)$, $f(x(t))$, and $f(v(t))$ to the exact free energy. In accordance with the discussion above, we see improved convergence for the x and v expansions. The expansion in v numerically appears to have a radius of convergence that extends to the entire physical range, zero to infinite temperature. Physics here does determine the range of convergence of normal-form theory.

The above results leave two questions: Can we improve our approximation for the free energy further, such that it converges, even at low orders, across all temperatures? Furthermore, can we approximate the coordinate change and analytic background without knowledge of the exact solution or the non-linear terms in the RG flows? To resolve these challenges, we again expand the free energy in v using the form given in Eq. (9), but fix the expansion coefficients of $\widetilde{t}$ and f_a by matching to low-temperature expansions of the free energy instead of expanding the exact solution. Importantly, this approach requires minimal knowledge about the critical point (only the asymptotic scaling form), with most of the information coming from deep within the low-temperature phase. Since the matching guarantees the correct low-temperature behavior and has the correct log singularity at the critical point built into the expansion, we see uniform convergence across all temperatures. For example, by sixth order, the approximation differs from the true free energy by at most 0.5% (red dashed line in Fig. 2(a)) and we see exponential convergence as we add additional terms (Fig. 2(b)).

Our approach for extending critical scaling forms to the neighboring phases should naturally generalize to unsolved statistical physics models and experimental systems. For example, low-temperature expansions are relatively easy to compute in all dimensions and for a variety of systems, and can be used to approximate the analytic corrections to established critical scaling forms. Candidates for this method include the 3D Ising model, where critical exponents are known to high precision from conformal bootstrap and the 2D Ising model in a field, whose scaling function is computed to high precision in Section 4. Finally, it is credible that the liquid phase, long known as being challenging because there is no 'small parameter,' could be described as a perturbation of the liquid–gas critical point, i.e. an Ising critical point plus analytic corrections (determined, for example, by matching to virial expansions).

4. 2D Ising Critical Point in a Field

Ever since Onsager solved the zero-field 2D Ising model, people have searched for a high-precision approximate solution for the 3D Ising model and occasionally also for the 2D Ising model in a field. The most successful of these attempts make use of parametric coordinates to interpolate behavior in temperature and field around the critical point in such a way that the singularity is naturally incorporated.[18,19] These coordinates, first introduced many decades ago,[20] are polar-like, with a radius-like component R that controls the proximity to the critical point and an angle-like coordinate θ that rotates around the temperature–field plane. When taken to parameterize the temperature–*magnetization* plane, such coordinates can be periodic in θ like true polar coordinates, something once studied by Michael and collaborators to describe the coexistence region.[21] However, when parameterizing the temperature–*field* plane, there is an inevitable cut resulting from the discontinuity between the up and down states at low temperatures, and θ is not a periodic coordinate but ends at the abrupt transition at some θ_0.

The power of adopting these coordinates is that, as a function of θ, the critical singularity does not appear, and one can reasonably well approximate the scaling function in this coordinate as a simple analytic function. Using this principle, Caselle *et al.* did a remarkably careful job of creating a function that satisfied lots of known properties and measured values (in yellow of Fig. 3). Their method was doomed to slow convergence at small external fields, though, because while the singularity of the critical point was indeed removed by the coordinate change, other more subtle singularities in the free energy remain, including a key essential singularity as one crosses the $h = 0$ line for $t < 0$. Figure 3 shows a comparison between the approximate form of Caselle *et al.* and that of our work including this singularity.

In order to rigorously realize the idea of a free energy scaling function broken into singular and analytic pieces, two singularities in the scaling function must be accounted for in addition to the critical one. One is ancient and well known to the experts: the Yang–Lee edge singularity for $T > T_c$ in the complex plane.[22,23] The other has only recently been realized to be relevant, but has a nice physical picture. As one crosses $h = 0$, the equilibrium magnetization jumps. But if you cross just a little bit, the up-spin magnetized state is metastable, lasting for a good while until a bubble of down spins forms and grows to flip the system. (Think of

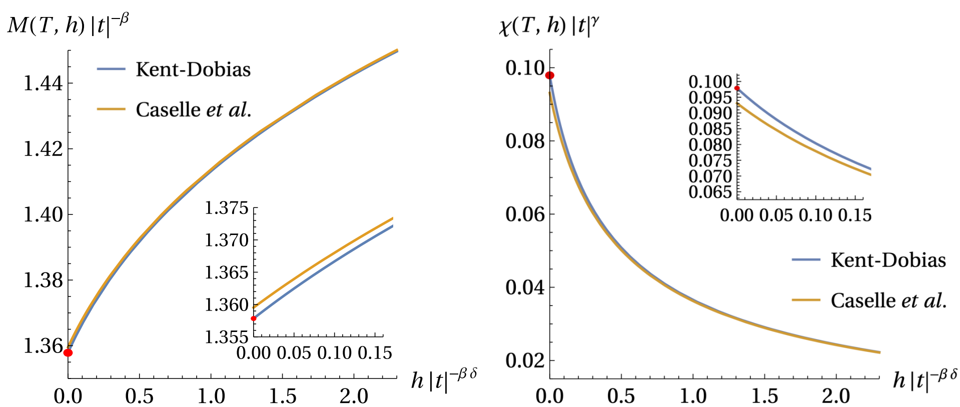

Fig. 3. Comparison between parametric approximations to the scaling functions with and without the appropriate singularities (from [6]). Scaling functions for the magnetization and susceptibility plotted as functions of the scaling invariant $h|t|^{-\beta\delta}$. The errors in the blue curves (Kent-Dobias) are estimated to be roughly 10^{-7}. Caselle's earlier work[19] that we build upon has significant errors near $h = 0$ (red point); these are due to the essential singularity at $h = 0$ and $t < 0$ that we address. These discrepancies grow larger with higher derivatives of the free energy, as shown in the right panel of Fig. 5.

supersaturated vapor — 101% humidity — and the nucleation of raindrops.) Here, the surface tension cost between the up and down regions for small droplets is bigger than the bulk free energy gain of aligning the interior with the external field. The energy barrier B to reach large sizes diverges as h vanishes, $B/k_B T = c/h$ for some constant c. Just like in quantum mechanics, where the lifetime of a state gives the energy an imaginary part, the free energy becomes complex for $h < 0$ with an essential singularity $\mathrm{Im} f \sim e^{(c/h)}$. One can use a Kramers–Kronig transform to see that the real part also has an essential singularity, influencing f for $h > 0$ as well.

We can incorporate these two singularities into the universal scaling function by first generating a 'simplest' functional form that has the correct singularities and then changing variables $\theta \to \widetilde{\theta} = g(\theta)$ and adding an overall analytic function $\mathcal{F}_a(\theta) = G(\theta)$, in a precise analogy with the analytic corrections we introduce to match the universal scaling function to describe the surrounding phases (Section 3).

The 'simplest' form is taken from writing the most singular part of the imaginary part of the free energy and then taking advantage of a Kramers–Kronig-like relation to find the corresponding real part. The requisite contour integral in the complex-θ plane is shown in Fig. 4.

Though complicated, this procedure pays dividends. By incorporating these singularities and matching the power series terms in $g(\theta)$ and $G(\theta)$ (analogous to $\widetilde{t}(t)$ and $f_a(t)$ in section 3), we are able to achieve exponential convergence of the scaling function to exactly known values at $t = 0$ and also achieve exponential convergence of derivatives. The left-hand side of Fig. 5 shows this convergence in the free energy itself as a function of the scaling invariant $th^{-\beta\delta}$ by subtracting our best converged approximation $\mathcal{F}^{[6]}$ from the lower-order approximations $\mathcal{F}^{[n]}(\theta)$ for $n \in \{2, 3, 4, 5\}$. These provide evidence that our seven-digit convergence at $t = 0$ extends to the whole scaling function. On the right of this plot, we can see the origin of this good behavior: The series expansion for the free energy around the abrupt transition point has zero radius of convergence, but this is captured naturally by our approximate scaling form.

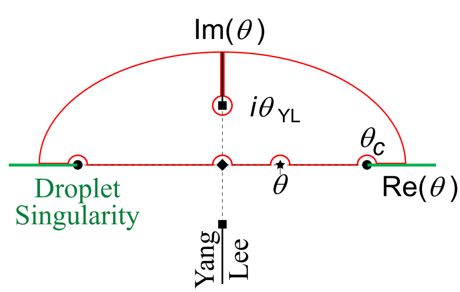

Fig. 4. Contour in the complex θ plane to generate a free energy scaling form with the correct singularities (from Ref. 6).

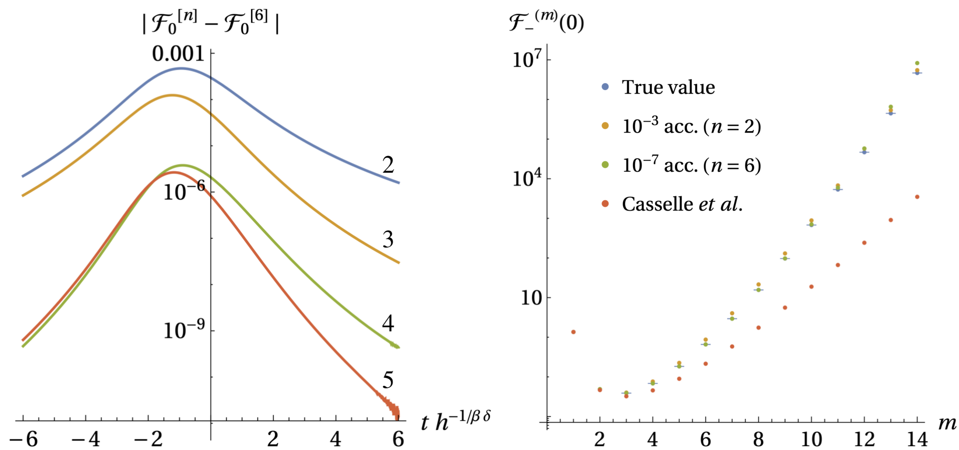

Fig. 5. Convergence of the universal scaling function (from Ref. 6). Left: The difference between the nth order approximation for the free energy scaling function in a field and the 6th order approximation as a function of the scaling invariant $th^{-1/\beta\delta}$. The 5th and 6th order approximations differ by at most 2×10^{-6}. Higher derivatives behave similarly. Right: The Taylor series coefficients $\mathcal{F}_-^{(m)}(0)$ of the free energy scaling function at the abrupt transition line (the location of the essential singularity) as a function of derivative m. By incorporating the essential singularity into the scaling function explicitly, the approximate forms have zero radius of convergence, matching numeric measurements of these coefficients.

5. Interpolating Scaling Functions Between Dimensions

Over five decades, the traditional thermodynamic critical points, illustrated in Fig. 6, have been extensively explored using ϵ-expansions, $1/N$ expansions, cluster expansions, etc. The universal critical exponents are known essentially exactly from conformal bootstrap methods.[c] Can we do the same for the universal scaling functions as functions of dimension d and the number of spin components N? And for (say) the random-field Ising model[24] or turbulence[25]?

Interpolation between dimensions could have several benefits. First, much is known analytically in two and four dimensions about the universal scaling functions that is only known numerically in three. Second, there are important features in scaling functions and their corrections to scaling that are a clear foreshadowing of properties in other dimensions. So, the leading correction to scaling in 3D is the echo of the marginal variable of the Wilson–Fisher transcritical bifurcation in 4D. And, in a more dramatic example, the universal scaling function for the avalanche size distribution in the 3D random-field Ising model has a striking feature (it grows

[c]In the same spirit, Kent-Dobias' solution[6] for the scaling function in 2D in Section 4 is also essentially exact: Both are well-defined algorithms that generate an exponentially converging approximation in a form useful for applications. After all, this is what we call the 'exact' solution for $\sin(x)$.

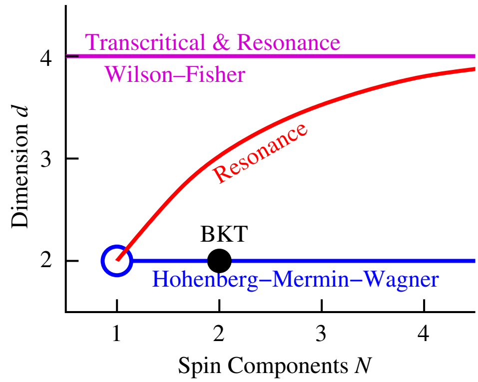

Fig. 6. Interpolating between dimensions: A schematic diagram showing the dimensions and spin components where traditional power-law scaling will break down. This leads to logarithms and exponentials replacing power laws and rich but complicated invariant scaling combinations as arguments for the universal scaling functions.[5] On the other hand, these complexities also promise to inform and accelerate convergence of the universal scaling functions for dimensions in between.

by a factor of 10 before cutting off exponentially[16]), which is clearly related to the unusual scaling of the avalanche sizes in two dimensions[15] (Fig. 1).

In ordinary differential equations, normal-form theory provides not only the behavior at the transition but also an unfolding of the behavior near the bifurcation (up to a smooth coordinate transformation). Here, we may expect to use this, not only to describe the phases near the critical points but also (following the lead of Wilson and Fisher) to describe the universal scaling functions as they evolve between dimensions.

How would we interpolate scaling functions, when even the arguments of the scaling functions vary with dimension (as in the W-functions we needed in Section 2 for the 4D Ising model)? Consider taking the equations for the 4D flow about the mean field fixed-point, Eq. (5), and keeping the unremovable non-linear terms to describe non-linear flows about the mean-field fixed point in d dimensions:

$$d\widetilde{f}/d\ell = d\widetilde{f} - \widetilde{t}^2, \quad d\widetilde{t}/d\ell = 2\widetilde{t} - A\widetilde{u}\widetilde{t}, \quad d\widetilde{u}/d\ell = (4-d)\widetilde{u} - \widetilde{u}^2 + D\widetilde{u}^3, \tag{12}$$

Things to note are as follows: (1) The normal-form procedure of removing the non-linear terms allows us to set the non-linear terms in any way we wish. (2) We know the Wilson–Fisher fixed point for Eq. (12) at $(\widetilde{t}^*, \widetilde{u}^*)$ must have a linearization with eigenvalues $1/\nu$ and ω that, as a function of dimension, match the conformal bootstrap values (or the exact values in 2D). This constrains the two non-linear terms $A(d)$ and $D(d)$. Serendipitously, the $\widetilde{t}^2$ resonance term in the 4D free energy flow is also needed to get the logarithmic specific heat in 2D. A high-precision universal scaling function for the free energy using the non-linear scaling variables $\widetilde{f}$, $\widetilde{t}$ and $\widetilde{u}$ in Eq. (12) would describe the universal crossover scaling from mean-field to short-range magnetization in all dimensions between two and four.

One could then linearize that scaling function about the 3D Wilson–Fisher fixed point to extract the traditional scaling functions.

How would one calculate such scaling functions? Here, serendipity strikes in recent advances in non-perturbative functional renormalization-group (NPFRG) calculations,[8] whose critical exponents[26] are almost competitive with those of conformal bootstrap, but which, in the process, also compute a functional form for the coarse-grained free energy as a function of magnetization.

Matthew Tissier and colleagues at the Jussieu campus of the Sorbonne and the first author have been exploring how to calculate high-precision universal scaling functions by combining the strengths of the normal-form theory with the systematically improvable scaling functions of NPFRG. In initial work, we have found that the NPFRG for the 4D Ising model should indeed have a scaling function whose arguments are of the exotic form in Eq. (6). We are now learning to solve the partial differential equations for $\widetilde{t}(t,u)$ $\widetilde{u}(t,u)$ and $\widetilde{f}(t,u)$ in 3D to extract the universal scaling functions from the simplest of NPFRG models. We hope then to use the technology in Section 4 to demonstrate how to tabulate universal scaling functions in a form convenient for experimentalists and simulators.

The NPFRG method has had striking success in high-precision calculations of equilibrium thermodynamic systems, disordered thermodynamic and avalanche models, fully developed turbulence, quantum many-body systems, and in QCD and electroweak models.[8] Our hope and vision is to inspire our colleagues to gather and tabulate their results into universal scaling functions that can describe the behaviors both near and far from points of emergent scale invariance in a way accessible to engineers, biologists, and social scientists studying complex systems.

6. Prospects and Challenges for Future Work

We have summarized what we believe to be promising indications that we can indeed generate usable, high-precision universal scaling functions in a systematic way and can extend them systematically into high-precision descriptions of the surrounding phases. If this continues to work, one imagines our community will be able to provide quantitative theories of liquids, material plasticity and failure, fluctuations in biological systems, and potentially turbulence and glass behavior. In all these cases, we must mesh our understanding of the universal, emergent critical behavior and our system-dependent knowledge of properties away from criticality to describe the entire phases. Will this work in practice?

To show that our method works to high precision, we chose to use a problem where we knew a great deal about the answer: the 2D Ising model. The reader should legitimately be skeptical that this is the easiest case. In particular, the 2D Ising mode is special in that it (1) is self-dual, (2) is analytically solvable in zero field, and (3) has no singular corrections to scaling. Let us discuss causes for concern that our methods will become general tools and reasons for optimism:

(1) *The 2D Ising model is self-dual.* In Section 3, we found exponential convergence of the Ising free energy $f(t)$ in zero field using a power series of the Onsager solution about the critical point. To do so, we expanded in a variable $v(t)$ that was fine-tuned to the self-dual and complex analytic properties of the 2D Ising model. We then used the low-temperature expansion to avoid any use of Onsager's solution. Will we need to find the 'right' variable in 3D to proceed? It turns out that matching to properties outside the radius of convergence stabilizes the expansion. In the preliminary work, matching both high- and low-temperature series using the standard variable $x = X - X_c = e^{-2/T} - e^{-2/T_c}$ in 2D (ignoring duality), we found exponential convergence in the entire temperature region.[4]

(2) *The 2D Ising model in zero field has an exact solution* and also has an exact solution at T_c as a function of field.[d] But will the external field prevent us from finding an exponentially converging solution using analytic corrections to scaling $\widetilde{t}(t, h)$, $\widetilde{h}(t, h)$, and $f_a(t, h)$, by matching the 2D universal scaling function from Section 4 to the low-T/high-h and high-T/low-h cluster expansions, as we did for zero field in Section 3?

We have an exact solution for the magnetization — the first derivative with respect to field. This has allowed us to determine $\widetilde{t}$ and $\widetilde{h}$ to linear order in h. The susceptibility $\chi(T)$ at zero field is not known exactly, but a remarkable amount is known: there are formal expansions strongly indicating subdominant logarithmic corrections to scaling and a possible essential boundary in the complex plane at the Fisher circle of zeros in the complex temperature plane.[27–30] Log corrections to scaling are known to arise from resonances between relevant and irrelevant eigenvalues (Section 2); the observed logs could arise from irrelevant variables whose resonances contribute only in $O(h^2)$ to the free energy. It appears that we can use these results[27] not only to determine $\widetilde{t}$ and $\widetilde{h}$ to quadratic order in h, but also to extract information about irrelevant variables and their singular corrections to scaling in 2D (see below).

The universal scaling function in 3D presumably should succumb to the same kind of systematic approximation that we used in Section 4 for 2D. All the ingredients appear to be available.[e] The Yang–Lee singularity in 3D[33] has yielded to a high–precision NPFRG calculation. The essential singularity calculation depends only on the value of the surface tension near the critical point. In 2D, we used estimates of derivatives of the universal scaling function near $h = 0$ for $T > T_c$ and $T < T_c$. The latter (shown in Fig. 5(b)) are challenging to calculate using NPFRG methods (perhaps because of the essential singularity), but the former are now available.[34] Indeed, these authors used their estimates and the traditional

[d] It may seem in retrospect clear that the essential singularity at $h = 0$ would frustrate a search for an exact solution in a field below T_c.

[e] And, of course, high-precision NPFRG calculations of the critical exponents[26] implicitly have calculated the universal scaling function.

Schofield coordinates to estimate the scaling function; adding the proper droplet singularity and Yang–Lee singularities could be straightforward and hopefully will yield convincing exponential convergence.

Can we expect to extend a high-precision 3D Ising scaling function to describe the in-field behavior in the entire surrounding phases? Using analytic corrections to extend the critical singularity could break down at the roughening transition.[f] However, this subtle transition should in principle also impede the use of Dlog Padé methods using the low-temperature expansion of the magnetization to describe the critical-point behavior, which works well in practice. In any case, the roughening transition does not arise in isotropic systems, so these concerns will not interfere with finally extracting a quantitative theory of liquids by adiabatic continuation of our understanding of the 3D Ising critical point.

(3) *The Onsager solution exhibits no singular corrections to scaling.* First, this statement is more subtle than it seems. Onsager's solution has a pure logarithmic singularity, but conceivably there could be irrelevant eigenvalues with integer exponents that would lead to analytic terms in the free energy and magnetization. Also, there are irrelevant anisotropies in the correlation functions and the susceptibilities due to the square lattice, indeed with integer exponents. Even more subtle, these anisotropies are singular corrections to scaling in Wilson's RG formulation, but are clearly present at the fixed point of the real-space RG. Any coarse-graining procedure maintaining a square lattice will have a short-range square anisotropy.

Experts will remember that some of the irrelevant operators in the RG are *redundant*, and these redundant operators often involve flows between different fixed points in the same universality class.[g] The idea of redundant variables had been first explored by Wegner,[38] who pointed out that a RG transformation would always have certain operators that are redundant; they correspond to infinitesimal redefinitions of the field and do not contribute to the singular part of the free energy.[h]

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[f] While one might argue that the behavior of an interface between spin-up and spin-down regions should not affect the bulk free energy, the (incredibly subtle) essential singularity as one crosses $h = 0$ (Section 4) depends on the surface energy of the critical droplet, which will itself have an essential singularity as a function of temperature as it develops facets at the roughening temperature.

[g] We have recently looked at this question[35] in the context of the period doubling onset of chaos in iterated maps $f(x)$. About half of the irrelevant eigenvalues are negative powers of the relevant critical exponent α and are redundant: One can set the amplitudes of their correction to scaling to zero using a coordinate change $x = \phi(y)$, so instead of $\alpha f(f(x/\alpha))$, one considers the renormalization group $\alpha\phi(f(f(\phi^{-1}(x)/\alpha))$. We called these *gauge irrelevant* as they depend on the choice of coordinate. A similar gauge invariance arises for the quasiperiodic onset of chaos[36,37] that we studied when the first author was a postdoc.

[h] How does one think of these redundant operators in the discrete theory? Murthy and Shankar had found a clever way to think about redundant operators for the discrete Ising model.[39] They essentially used a version of Kadanoff's RG projection operator without reducing the degrees of freedom. They found evidence that these redundant operators had observable consequences in simulations of the 2D Ising model.[40] Evidence for redundant operators had already been found in the 3D Ising model.[41]

The first author's first graduate student, Mohit Randeria, worked for Michael Fisher before shifting to his group. Randeria and Fisher wrote a comment on a paper by Swendsen[42] which claimed that one could modify the RG transformation to move the fixed point anywhere on the critical surface. They asserted that this could not in general be correct because the amplitudes of the singular corrections to scaling are zero at the fixed points. To quote them:

> "One fixed point may be mapped into another by a change of redundant variables and two RGs, say $\mathcal{R}$ and $\widehat{\mathcal{R}}$, may produce *formally* different fixed points by this mechanism. Nevertheless, a general point on a critical manifold *cannot* be transformed into a fixed point nor *vice versa*."

In the context of the 2D Ising model, there seem to be no singular corrections in the exact solution. Barma and Fisher had predicted that there should be logarithmic corrections to scaling coming from irrelevant variables.[43] They had also found evidence for an irrelevant operator with exponent $-4/3$.[44] Conformal field theory predicts a large number of irrelevant variables which come from descendant operators but does not predict the $-4/3$ exponent. It has been conjectured in the past that all of these descendent operators are redundant.[45] This may explain why they do not lead to any logarithmic corrections to scaling (which otherwise are generically expected[14]). We would consider all of these descendant operators as constituting 'gauge' corrections to scaling, which can be removed by an appropriate coordinate choice, whereas the $-4/3$ exponent would be a genuine singular correction to scaling.

In Section 3, we succeeded in solving for the free energy of the square-lattice Ising model at all temperatures by applying a normal-form change of variables to the universal scaling function for the critical point. Is that success due to the lack of (non-redundant) singular corrections to scaling for this particular universality class?[i] It is clear that in 3D, we shall need to incorporate the (genuine) singular corrections to scaling in order to accurately describe the behavior away from the fixed point. Indeed, these contributions are subdominant near the fixed point, meaning conversely that they grow faster as one leaves the fixed point than the relevant operators. Thus, we will need a universal scaling function that depends not only on $\widetilde{t}$ and $\widetilde{h}$ but also on potentially an infinite family of irrelevant variables $\widetilde{u}$. Can we nonetheless hope for exponential accuracy?

The NPFRG methods mentioned in Section 5, as it happens, precisely implement a flow with many irrelevant eigendirections at the fixed point. Applying our normal-form analysis, we have been able to linearize this flow in the simplest case, allowing us to extract a functional form for the leading correction subdominant as $\widetilde{t}^{\omega\nu}$. It

[i]It is believed that the powers of logarithms in the 2D Ising susceptibility vanish in isotropic systems. Since all irrelevant operators in 2D have rational eigenvalues, and the redundant ones cannot generate resonant logarithms, one suspects that all the isotropic corrections to scaling are redundant.

seems likely that one could generate exponential convergence for all properties of the phases by combining increasingly sophisticated NPFRG calculations, adding more irrelevant operators to our scaling functions, and matching to higher orders of low- and high-temperature expansions (or virial expansions, $1/N$ expansions, etc.).

Finally, there are intriguing hints of deep connections between normal-form theory, redundant operators, and the construction of universal scaling functions. We noted in Section 4 that the procedure we used to generate the universal scaling function recapitulated exactly the same steps as the procedure we used in Section 3 to generate the Onsager solution from the critical point. In both cases, we knew the singularities of the functions involved and fit parameters in two functions (an analytic change of variables and an additive analytic background) to match known properties measured separately. If this systematic procedure can be generally used as a best practice, rapid progress could be made.

There is also a striking analogy between redundant operators (removing some singular corrections to scaling by reparameterizing the space of predictions) and our normal-form theory (removing analytic corrections to scaling by reparameterizing the control parameters). Can we simultaneously remove both with a joint transformation? How are they related?

In thermodynamics, whether you write the free energies as a function of the external field and temperature or as a function of magnetization and temperature is considered a matter of choice. A Legendre transformation from control parameters (f, t, h) to (u, t, m) swaps the control parameter h with the prediction m: Does it swap normal-form transformations to redundant ones? (Indeed, the NPFRG methods coarse-grain the fixed-magnetization Gibbs free energy $u(t, m)$ rather than Wilson's free energy at fixed field.) When we use our normal-form theory to remove non-linear terms in the RG flows by changing h to $\widetilde{h}(t, h)$, we change the predicted magnetization. Is that change a redundant one?

Hankey and Stanley[46] showed that if you Legendre transform the generalized homogeneous functions we use for hyperbolic fixed points, you get another homogeneous function. Later, others, studying finite-size scaling of the Ising model[47,48] in the microcanonical ensemble predicted that it would be possible to speak of the entropy as a scaling function of the energy instead of the free energy as a function of the temperature.

Our preliminary work on this question ran into a difficulty, generalizing these ideas to the non-linear normal forms necessary in the upper and lower critical dimensions. We have found that it is possible to Legendre transform RG flows at the linear level, but non-linear flows generically lead to non-analyticities in the RG flow.[14,32,i] Whether this clarifies or confuses the correspondence between redundant and normal-form changes of variables needs to be explored.

[i] As noted in the footnote on page 418, even in four dimensions, the RG flow of the magnetization is linearizable. Presumably, this means that implementing an RG using flows in (f, t, h) and (u, t, m) will be possible even in four dimensions, while we find (f, t, h) to (S, E, h) leads to non-analytic RG flows.

As a long-time colleague of Michael Fisher, the lead author hopes that Fisher would have enjoyed this deep plunge into the complex history of the field and our ambitions for the future. He also hopes that Fisher's friends, colleagues, and collaborators who are contributing to this book, and of course our colleagues of the future who are reading it, will find this chapter more illuminating than obscure, and more useful than misleading or misguided.

Acknowledgments

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