Fracture Strength of Disordered Media: Universality, Interactions, and Tail Asymptotics

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We study the asymptotic properties of fracture strength distributions of disordered elastic media by a combination of renormalization group, extreme value theory, and numerical simulation. We investigate the validity of the ‘‘weakest-link hypothesis’’ in the presence of realistic long-ranged interactions in the random fuse model. Numerical simulations indicate that the fracture strength is well-described by the Duxbury-Leath-Beale (DLB) distribution which is shown to flow asymptotically to the Gumbel distribution. We explore the relation between the extreme value distributions and the DLB-type asymptotic distributions and show that the universal extreme value forms may not be appropriate to describe the nonuniversal low-strength tail.

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It has been known for centuries that larger bodies have a lower fracture strength. The traditional explanation of this size effect is the ‘‘weakest-link hypothesis’’: the sample is envisaged as a set of noninteracting subvolumes with different failure thresholds, and its strength is determined by the failure of the weakest region. If the subvolume threshold distribution has a power-law tail near zero, then the strength distribution can be shown to converge to the universal Weibull distribution for large sample sizes [1], an early application of extreme value theory (EVT) [2].

Often failure occurs due to the presence and growth of microcracks whose long-range interactions call the notion of independent subvolumes into question. There have been two broad approaches to address such interactions: fiber bundle models and fracture network models [3]. Fiber bundles transfer loads by various rules as individual fibers fail; in some particular cases, exact asymptotic results for the failure distribution have been derived [4] and do not explicitly fall into any of the extreme value statistics universal forms. Fracture network models consider networks of elastic elements with realistic long-range interactions and disorder. A particularly simple approach is based on the random fuse model (RFM) [3,5], where one approximates continuum elasticity with a discretized scalar representation. It has been suggested that, in the weak disorder limit, fracture would be ruled by the longest microcrack present in the system [6–9]. By using critical droplet-theory-type arguments, one can show that an exponential distribution of microcracks leads to the Duxbury-Leath-Beale (DLB) distribution of failure strengths [7], which again does not explicitly have an extreme value form.

These studies raise three important questions. First, what is the importance of elastic interactions in determining the strength distributions, and does the weakest-link hypothesis hold in the presence of such interactions? Second, what is the relation between the DLB-type asymptotic strength distributions and the universal forms predicted by EVT? Third, how should one best extrapolate from measured strength distributions to predict the probability of rare catastrophic events? We use renormalization group (RG) ideas, EVT, and simulations of the two-dimensional RFM to explore these questions. We conclude that (i) the weakest-link hypothesis is valid for large samples even in the presence of long-ranged elastic interactions; (ii) the asymptotic forms of the strength distribution for these interacting models are compatible in disguise with EVT, but of the Gumbel form rather than the Weibull form; and (iii) the use of extreme value distributions to estimate the probability of rare events, although common in the experimental literature, is not always justified theoretically. DLB-type asymptotic distributions (or those derived by Phoenix [4]) which depend on the details of the material are necessary to safely extrapolate deep into the tails of the failure distribution.

The RG and the EVT present two equivalent, yet contrasting, approaches to the study of the universal aspects of extreme value distributions, in general [10], and fracture strengths, in particular. The natural framework to investigate the role of interactions and the corrections to scaling that emerge as the system size is changed is provided by the RG theory. In contrast, the EVT facilitates the study of domains of attraction and convergence issues. The nonuniversal, yet important, behavior of the low-reliability tail of the distribution is not described adequately by either the RG or the EVT. To study such nonuniversal features, one needs to develop DLB-type asymptotic theories.

Typically, a RG transformation proceeds in two steps: in the first step, the system is coarse-grained by eliminating short length-scale degrees of freedom, and then the resulting system is rescaled. The RG coarse graining for fracture is equivalent to the weakest-link hypothesis: a system of size L in $d = 2$ dimensions survives at a stress σ if its 4 ($= 2^d$) subsystems of size $L/2$ survive at the same stress. This coarse graining leads to the following recursion relation for $S_L(\sigma)$—the probability that a system of size L does not fail under a stress σ:

$$S_L(\sigma) = [S_{L/2}(\sigma)]^4. \tag{1}$$

The second step of the RG transformation is to rescale the stress suitably and look for a fixed point distribution S^* that is invariant under RG,

$$S^*(\sigma) = \mathcal{R}[S^*(\sigma)] = [S^*(a\sigma + b)]^4. \tag{2}$$

Instead of applying Eq. (1) iteratively like the RG, the EVT formulation considers the large length-scale limit directly,

$$S^*(\sigma) = \lim_{L\to\infty} [S_{L_0}(A_L\sigma + B_L)]^{(L/L_0)^d}, \tag{3}$$

where L_0 is a characteristic length scale.

The functional Eqs. (2) and (3) are known to have only three solutions: the Gumbel, the Weibull, and the Fréchet distributions. Of these, only the Gumbel $\{S^*(\sigma) = \Lambda(\sigma) \equiv \exp[-e^{\sigma}],\ \sigma \in \Re,\ a = 1,\ b = \log 4\}$ and the Weibull $[S^*(\sigma) = \Psi_\alpha(\sigma) \equiv e^{-\sigma^\alpha},\ \sigma,\ \alpha > 0,\ a = 4^{(-1/\alpha)},\ b = 0]$ distributions are relevant for fracture. The large length-norming constants, A_L and B_L, satisfy the following asymptotic relations, $A_{2L}/A_L \to 1/a$ and $|B_{2L} - B_L|/A_L \to b/a$.

To test the validity of the weakest-link hypothesis [Eq. (1)] in the presence of long-range elastic interactions, we perform large-scale simulations of the RFM [3,5], considering a tilted square lattice (diamond lattice) with $L \times L$ bonds of unit conductance. Initially, we remove a fraction $1 - p$ of the fuses at random, where p is varied between $1 - p = 0.05$ and $1 - p = 0.35$ (the percolation threshold for this model is at $p = 1/2$). Periodic boundary conditions are imposed in the horizontal direction, and a constant voltage difference, V, is applied between the top and the bottom of the lattice system bus bars. The Kirchhoff equations are solved to determine the current distribution on the lattice. A fuse breaks irreversibly whenever the local current exceeds a threshold that we set to one. Each time a fuse is broken, we recalculate the currents in the lattice and find the next fuse to break. The process is repeated until the system is disconnected. In the present simulations, we have considered system sizes from $L = 16$ to $L = 1024$ and various values of p. To explore the low-strength tail which is beyond the accessible range of most experiments, we typically average our results over 10^5 realizations of the initial disorder. The fuse model is equivalent to a scalar elastic problem. Using this equivalence, the strain is defined as $\epsilon = V/L$ and the stress is given by $\sigma = I/L$, where I is the current flowing in the lattice. The fracture strength is defined as the maximum value of σ during the simulation.

The RG coarse-graining step [Eq. (1)] produces a natural test for the weakest-link hypothesis. In Fig. 1, we report the survival probability $S_L(\sigma)$ for different system sizes L, compared with those for systems of size $L/2$, rescaled according to Eq. (1). The agreement between the two distributions is almost perfect for $L/2 \geq 32$, indicating that Eq. (1) is satisfied asymptotically. Corrections to scaling due to the effect of distant microcracks are expected to decay as $1/L^2$, as can be shown by a direct calculation, but are too small for us to detect in simulations (Fig. 1). We also tested wide rectangular systems with $L_x = 2L_y$, finding larger corrections, scaling roughly as $1/L$, which are still irrelevant in the large system size limit.

Duxbury *et al.* related the survival distribution to the distribution of microcrack widths w [7]. At the beginning of the simulation, the "per-site" probability distribution of a crack of width w is $P(w < w') = 1 - e^{-w'/w_0}$, where $w_0 \sim -1/\log 2(1 - p)$ [11]. Hence, the distribution of the longest crack, w_m, in a lattice with L^2 sites, is given by

$$P(w_m < w') = (1 - e^{-w'/w_0})^{L^2}. \tag{4}$$

The stress at the tip of a crack of width w is asymptotic to $\sigma K\sqrt{w}$, where σ is the applied far-field stress and K is a lattice-dependent constant. A sample survives until the largest crack becomes unstable when its tip stress reaches a threshold $\sigma_{\rm th} = \sigma K\sqrt{w}$. Therefore, we have

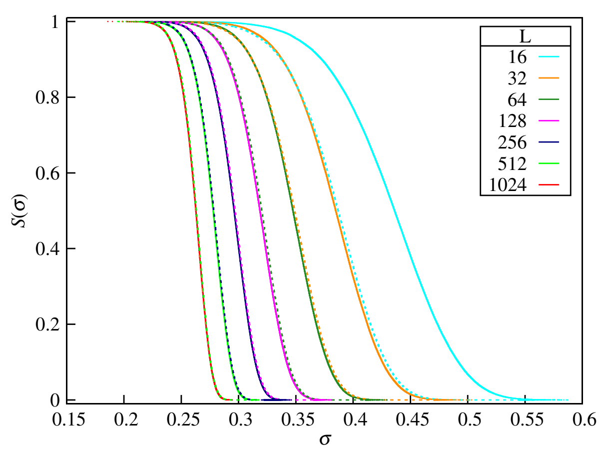

FIG. 1 (color online). Testing the weakest-link hypothesis by comparing the survival probability $S_L(\sigma)$ for a $L \times L$ network (solid lines) with that predicted by the weakest-link hypothesis, $S_{L/2}(\sigma)^4$ (dotted lines), for $1 - p = 0.10$. Note the excellent agreement even for moderate system sizes.

$$S_L(\sigma) \simeq \left(1 - e^{-(\sigma_0/\sigma)}\right)^{L^2} \simeq D_L(\sigma), \tag{5}$$

where $\sigma_0 \equiv \sigma_{\text{th}}/K\sqrt{w_0}$ and $D_L(\sigma) \equiv \exp[-L^2 e^{-(\sigma_0/\sigma)^2}]$ is the DLB distribution. To apply the above derivation to the failure stress, we first check the distribution of microcrack lengths at peak load. As shown in Fig. 2, the distribution is exponential, but, due to damage accumulation, the slope of the tail changes with respect to the initial distribution. This appears to be due to bridging events in which two neighboring cracks join, leading to a modification of Eq. (5), as discussed in Ref. [7]. Thus, damage accumulation, although very small, is relevant because it changes the exponent of the microcrack distribution. The exponential form of the crack length distribution tail, however, suggests that the DLB form should still be valid, as demonstrated in Fig. 3. In particular, the average failure stress scales as $\langle\sigma\rangle = \sigma_0/\sqrt{\log(L^2)}$ [Fig. 3(a)] and the distributions for different L all collapse into a straight line when plotted in terms of rescaled coordinates [Fig. 3(b)].

Our arguments thus far are seemingly paradoxical. On the one hand, we have argued on very general grounds that the distribution of failure strengths must be either Gumbel or Weibull, while, on the other hand, we have checked that the failure distribution for fuse networks is of the rather different form proposed by Duxbury *et al*. How can this "paradox" be resolved? While it is not guaranteed that a microscopic survival distribution will lead to a fixed point under linear rescaling [Eqs. (2) and (3)], the DLB distribution does converge to the Gumbel form, i.e.,

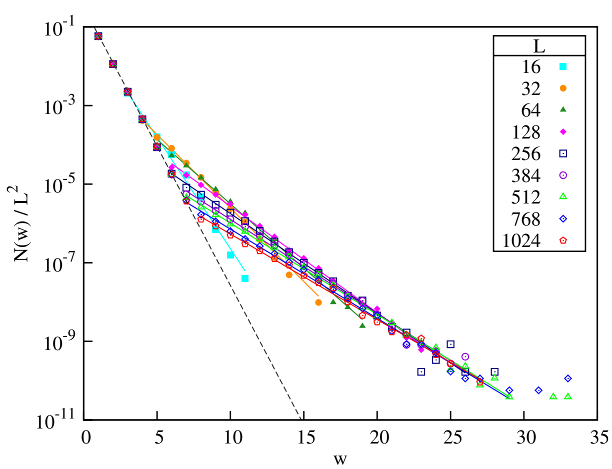

FIG. 2 (color online). Crack width distributions at peak load, $1 - p = 0.10$. The initial distribution of microcrack widths [$N(w)$ is the number of clusters of width w] is exponential with slope $\approx -\log 2(1-p)$ (dotted line). As the system is loaded, a few bonds break before catastrophic failure; these bonds usually connect smaller clusters, producing extra cracks at large widths. The resulting crack width distribution at the peak load exhibits a size-dependent crossover to a different exponential slope. Solid lines represent fits to an exponential.

$$\lim_{L\to\infty} D_L(A_L\sigma + B_L) = \Lambda(\sigma), \tag{6}$$

as can be demonstrated by a straightforward calculation using $A_L = \sigma_0/\{2[\log(L^2)]^{3/2}\}$ and $B_L = \sigma_0/\sqrt{\log(L^2)}$. The above result is striking because fracture distributions are usually assumed to not be of the Gumbel form, since fracture must happen at positive stress, while the Gumbel distribution has support for negative arguments, as well. This is akin to arguing that the normal distribution is not valid for test scores since scores must always be positive. Nonetheless, it brings us to the issue of convergence and validity of extreme value distributions as opposed to DLB-type distributions.

The extreme value distributions, $S^*(\sigma)$ [$= \Lambda(\sigma)$ or $\Psi_\alpha(\sigma)$], are a uniform approximation to the true survival function, $S_L(\sigma)$, for all σ in the limit of large L, i.e.,

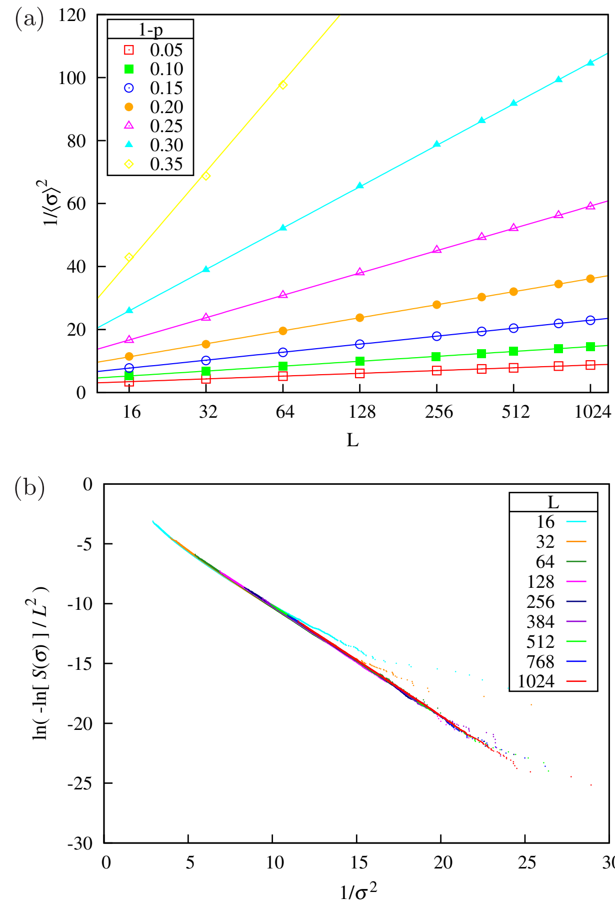

FIG. 3 (color online). Testing the DLB distribution of failure stresses. (a) The average failure stress as a function of system size L at various bond fractions p (symbols) can be fit well by the DLB form (solid lines), except close to the percolation threshold ($1 - p > 0.3$). (b) A collapse of the strength distribution for different system sizes at $1 - p = 0.1$, such that the DLB form would collapse onto a straight line.

$$\lim_{L\to\infty}\left[\sup_{\sigma\in\Re}\left|S_L(\sigma)-S^*\left(\frac{\sigma-B_L}{A_L}\right)\right|\right]=0. \tag{7}$$

In contrast, DLB-type distributions [12] are based on material details and are asymptotically correct in the low-reliability tail, i.e.,

$$\lim_{L\to\infty}\left(\lim_{\sigma\to 0}\frac{1-D_L(\sigma)}{1-S_L(\sigma)}\right)=1. \tag{8}$$

Note that the uniform convergence in Eq. (7) does not bound the *relative* error in the low-reliability tail, while the asymptotic convergence in Eq. (8) does.

The above discussion hints at an underlying question: How does one accurately predict the probability of rare small-strength events with limited experimental data? The standard practice is to measure the failure distribution of construction beams or microcircuit wires, fit to the universal Weibull or Gumbel form, and extrapolate. However, as we have argued, this approach can lead to incorrect estimates. The low-reliability tail is nonuniversal and must be modeled by a theory that, like DLB, accounts for microscopic details (see also [13]). Such theories, analogous to critical droplet theory (low temperatures), instantons (low $\hbar$), and Lifshitz tails (low disorder, deep in the band gap) are, by construction, accurate in the low-reliability tail. In this case, a fit to the Weibull or Gumbel form overestimates the low-stress failure probability and hence might be appropriate as a conservative estimate (e.g., construction beams) but not when optimizing a design (e.g., circuits). It is interesting to observe that usually the RG and the critical droplet theory address continuous and abrupt phase transitions, respectively; yet, here, these two approaches both apply to fracture.

The convergence to extreme value distributions can be extremely slow [13]. For the RFM, let z be the number of standard deviations up to which the Gumbel approximation is accurate within a relative error of ϵ. By using the Edgeworth-type expansions for the extreme value distributions [14], we find

$$\frac{z\pi}{\sqrt{6}}=\begin{cases}\sqrt{\eta}\exp\left\{-\frac{\sqrt{\eta}}{2}\exp\left[-\frac{\sqrt{\eta}}{2}\exp[\ldots]\right]\right\}, & \eta<4e^2\\ \log\eta-2\log\{\log\eta-2\log[\ldots]\}, & \eta>4e^2,\end{cases}$$

where the ellipses indicate an infinite recursion and $\eta=-(4/3)\log(1-\epsilon)\log(L^2)$. For an accuracy of 10% at 1 standard deviation, a sample volume of $L^2\approx 10^{18}$ is required, while, at 2 standard deviations, the required sample volume is about $L^2\approx 10^{264}$. As a comparison, for the Gaussian approximation to the mean of a sample of $M(\gg 1)$ random variables (normalized so that $E[X]=0$, $E[X^2]=1$, and $E[X^3]=\gamma$), we get $z\sim\Delta^{1/3}+\Delta^{-1/3}+\mathcal{O}(\Delta^{-4/3})$, where $\Delta=6\epsilon\sqrt{M}/\gamma$; thus, $z\approx 3$ for $\epsilon=0.1$, $M=3000$, and $\gamma=2$, where the value $\gamma=2$ corresponds to the standard exponential distribution. However, the universal extreme value forms are not always dangerous for extrapolation. One can show that they are valid asymptotic forms, in the manner of Eq. (8), if they satisfy the condition of tail equivalence ([15], p. 102, and [16]):

$$\lim_{\sigma\to 0}\frac{1-S_L(\sigma)}{1-S^*(\sigma)}=C, \qquad 0<C<\infty. \tag{9}$$

The success of the classical example of a Weibull distribution of failure strengths emerging from a power-law microcrack length distribution may be due to the tail equivalence of the microscopic and the Weibull distributions.

In conclusion, by using a combination of renormalization group, extreme value theory, and numerical simulations, we have shown that the failure strength of an elastic solid with a random distribution of microcracks follows the DLB distribution which asymptotically falls into the Gumbel universality class. The nonuniversal low-reliability tail of the strength distribution may not be described by the universal extreme value distributions, and thus the common practice of fitting experimental data to universal forms and extrapolating in the tails is questionable. Theories that account for microscopic mechanisms of failures [13]—the DLB distribution, for instance—are required for the accurate prediction of low-strength failures. In our study, the emergence of a Gumbel distribution of fracture strengths is surprising and brings into question the widespread use of the Weibull distribution for fitting experimental data.

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[11] The factor 2 in $w_0 \sim -1/\log 2(1-p)$ is due to the fact that, on the diamond lattice, there are 2^w crack "backbones" of width w with a given site as their left end.

[12] The DLB distribution is asymptotically exact in the limit of $1-p \rightarrow 0$.

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