MEISSNER EFFECTS, VORTEX CORE STATES, AND THE VORTEX GLASS PHASE TRANSITION

by

Ming Huang

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AND THE VORTEX GLASS
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A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by

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August 1991

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Ming Huang, Ph.D.
Cornell University 1991

This thesis covers three topics involving Meissner effects and the resulting defect structures. The first is a study of Meissner effects in superconductivity and in systems with broken translational symmetry. The Meissner effect in superconductors is a rigidity against external magnetic field caused by the breaking of the gauge symmetry. Other condensed matter systems also exhibits rigidities like this: The breaking of the translational symmetry in a cubic-liquid-crystal causes the system to expel twist deformations and the breaking of the translational symmetry in a nematic liquid crystal gives it a tendency to expel twist and bend deformations. In this thesis, we study these generalized Meissner effects in detail.

The second is a study of the quasiparticle states bound to the vortex defect in superconductors. Scanning-tunneling-microscope measurements by Harald Hess *et al.* of the local density of states in a vortex core show a pronounced peak at small bias. These measurements contradict with previous theoretical calculations. Here, we solve the Bogoliubov equations to obtain the local density of states in the core and satisfactorily explain the experimental observations. We also predicted additional structure in the local density of states which were later observed in experiments.

The third is a study of vortex dynamics in the precense of disorder. A mean field theory is developed for the recently proposed normal to superconducting vortex glass transition. Using techniques developed to study the critical dynamics of spin glasses, we calculate the mean field vortex glass phase boundary and the critical exponents. We also explain the experimentally observed magnetic field induced transition broadening.

Biographical Sketch

Ming Huang was born on March 29, 1964 in Wuhan, China. He lived there until he was 17. From 1981 to 1985 he attended Beijing University, where he met his future wife Qin Zhang and obtained a bachelor's degree in physics. Since 1985 he has been a graduate student in physics at Cornell University. Ten years of sweat and joy in physics did not prevent him from changing his career to finance. Starting this fall, he will attend Stanford Business School for yet another doctor's degree.

To my parents

Acknowledgments

I would like to thank my thesis advisor Jim Sethna. From him I learned how to understand complicated problems in simple terms and not to lose confidence when facing difficulties. His kindness, enthusiasm, patience, and encouragement made my study at Cornell a very enjoyable experience.

I would also like to thank Alan Dorsey for introducing me to the problem of the vortex glass transition. I benefited tremendously from our collaboration and I enjoyed working with him.

I also want to thank the following people for their support and for many enjoyable moments:

— to my roommate Zhenyu Duanmu for many interesting discussions about physics, politics, and other things;

— to Chris Myers for *tirelessly* answering many of my computer questions and for talking about baseballs with me;

— to Eric Grannan for many frustrating (*to me*) ping pong ball games;

— to Joanna Wong for spending hours helping me work through my first few computer programs;

— to Joel Shore for our collaboration on the vortex core states;

— to Xiaomei Song for making two nice figures for my thesis;

— to some other Sethna group members – Jean Carlson, Steve Langer, and Shelly Shumway – for many enjoyable group dinners and meetings;

— to all of my officemates – Alan Denton, Jean Carlson, Larry Shaw, Rob Smith, Bruce Roberts, and Stefan Brundobler – for many interesting discussions;

— to the New York Mets for many enjoyable baseball games when I didn't feel like doing physics;

— to the physics Softball Recreational League team for giving me the opportunities to drop many fly balls and to make a few unforgetable (*by me*) catches.

Last, but not the least, I want to thank my wife Qin Zhang for tolerating my erratic sleeping schedule and for her support through my graduate school years. I also want to thank my parents, who instilled in me a strong sense of importance of education although they themselves never experienced any formal education.

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Chapter 1

Introduction

This thesis covers three of the topics I worked on in collaboration with various people during my graduate study at Cornell. It involves Meissner effects and the resulting defect structures. We first study the Meissner effect in superconductivity and similar rigidities in other systems in condensed matter physics. We next study the quasiparticle states bound to the vortex defect in superconductors. We will calculate the local density of states in a vortex core in an effort to explain a recent scanning-tunneling-microscope measurement. Lastly, we use the vortex glass model to describe the mixed state properties of the high temperature superconductors and study the mean field critical properties of the normal to vortex glass phase transition.

1.1 Meissner Effects

The Meissner effect is perfect diamagnetism: A weak magnetic field cannot penetrate into the interior of a superconductor. This expulsion of magnetic field can also be seen as a rigidity caused by a broken gauge symmetry. Effects like this occur in other areas of condensed matter physics too. A cubic-liquid-crystal phase with a nearest neighbor bond orientational order but no translational order will develop a rigidity against rotational twist deformations as the translational symmetry is broken. The resulting crystal expels twist deformations from its interior. Similarly, a nematic liquid crystal with a molecular rotational orientational order will also tend to expel molecular rotational deformations (although not perfectly, as we shall see) when the liquid crystal further breaks the translational symmetry and becomes a smectic liquid crystal.

The mathematical formulations of these rigidities are analogous: A system with a certain order has an elasticity like free energy representing the cost of energy due to deformations of the order: $(\vec{\nabla} \times \vec{A})^2$ represents the magnetic energy in a normal metal, $(\vec{\nabla} \times \vec{\theta})^2$ and $(\vec{\nabla} \cdot \vec{\theta})^2$ represent the energy costs of rotational deformations of the orientational order of the nearest neighbor bonds in a cubic-liquid-crystal phase with bond orientation represented by a rotation angle $\vec{\theta}$, and $(\vec{\nabla} \times \vec{n})^2$ and $(\vec{\nabla} \cdot \vec{n})^2$ represent the energy costs of the molecular rotational deformations of the molecular orientational order, where $\vec{n}$ represents the molecular orientation. As the system breaks a symmetry (gauge symmetry for superconductors and translational symmetry for crystals and smectic liquid crystals), it develops a new free energy term describing the energy cost of deformations of the new order. This new term couples the new order parameter to the old order parameter in order to satisfy a general invariance principle (gauge invariance for superconductors and space rotational invariance for crystals and smectic liquid crystals). This coupling causes the system to be rigid with respect to deformations of the old order. Therefore, superconductors expel magnetic field, crystals expel twist deformations, and smectic liquid crystals have (as we shall see) an imperfect expulsion of twist and bend deformations. We generally refer to these effects as Meissner effects.

In chapter 2, we will study the Meissner effect in superconductors by comparing it to the analogous Higgs mechanism in particle physics. In chapter 3, we will study how the breaking of the translational symmetry in a cubic-liquid-crystal can cause the system to expel twist deformations. In chapter 4, we will study the rigidity of a smectic liquid crystal with respect to twist and bend deformations as a result of the broken translational symmetry and discuss some special properties of smectic liquid crystals that lead to the *failure* of complete expulsion of twist and bend.

1.2 Vortex Core States

According to the BCS theory of superconductivity, a homogeneous superconductor has an energy gap for excitations, and expels small external magnetic fields. If the superconductor is type II, a larger field will penetrate into the bulk of the superconductor as a series of vortex lines. This also affects the superconducting gap: a vortex is like a small normal region, and there are local excitations bound to it with small energies. Recently, Harald Hess *et al.* performed a scanning-tunneling-microscope measurement on a $NbSe_2$ superconductor and measured the local density of states of an isolated vortex. The measurement showed a large peak in the local density of states at zero bias at the center of the vortex — a larger density of states than one would find in a normal region. This result is different from a previous theoretical calculation, which gave local states but no peak.

In chapter 5, we will study this problem in detail and show that a careful calculation within BCS theory can explain the experimental measurement. We also predicted double peak structures in the local density of states away from the center of the vortex. This prediction was later observed in experiments.

1.3 Vortex Glass Phase Transition

Recently discovered high temperature superconductors have several unusual mixed state properties that include an "irreversibility" line in the $H - T$ phase diagram and a magnetic field induced broadening of the resistive transition. To explain these observations, Matthew P. A. Fisher proposed a vortex glass model to describe the mixed state properties of the high temperature superconductors. In the vortex glass phase, the vortices are frozen by impurities into a "spin glass" like order, and the linear resistivity is zero. In chapter 6, we study the normal to vortex glass phase transition. Within a mean field theory, we calculate the vortex glass phase boundary, the mean field critical exponents, and the vortex glass order parameter susceptibility. We will also explain the magnetic field induced transition broadening within this model.

Chapter 2

The Higgs Mechanism and Superconductivity

2.1 Introduction

The phenomenon of symmetry breaking occurs when the ground state of a system does not possess the full symmetry of the Hamiltonian. It is very frequently encountered in condensed matter physics : the crystalline lattice breaks translational symmetry; the Heisenberg ferromagnetic ground state breaks rotational symmetry; superfluidity (including superconductivity as superfluidity with charged fluids) breaks gauge symmetry, and so on. In quantum field theories, however, since one does not know beforehand what the full symmetry of the laws of nature is, it is not clear whether there exists any broken symmetry. On the other hand, we are curious to know if all the symmetries of the laws of nature have been revealed to us through vacuum states (and their excitations – particles) that do not break any symmetry of the laws of nature, or if there exists some hidden symmetries that we do not know of since the vacuum states have broken them.

In quantum field theory, Goldstone *et al.*[1,2] have shown, at first through a model Lagrangian of a complex boson field with U(1) symmetry and then through a general theorem, that an ordinary* field theory with a vacuum state breaking a continuous internal symmetry necessarily implies the existence of a massless spinless boson (the Goldstone boson). One might simply conclude that no symmetry is broken from the fact that no massless spinless boson has been observed. That

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* Meaning a field theory obeying the usual axioms (Lorentz invariance, locality, Hilbert space with positive-definite inner product, etc.)

this simple conclusion is wrong was pointed out by Higgs[3] who, as shown later, extended the U(1) symmetry group of the Lagrangian of two scalar fields studied by Goldstone to a local U(1) symmetry group (local gauge invariance) by coupling a vector gauge field of photons to the scalar fields. By doing this, he was able to show that the Goldstone boson and the photon field degrees of freedom combined to give a massive vector meson —the photon acquires a mass. No Goldstone boson is implied by a broken continuous symmetry in this theory. This process of gauging away the Goldstone boson and of the photon becoming heavy is called the Higgs mechanism. The Higgs mechanism is important in quantum field theory because it can be used to construct theories that have broken symmetries but do not have Goldstone bosons (which are not observed in the real world of high energy physics).

In condensed matter physics, as we will show later, the Higgs mechanism can be used to explain how a superconductivity theory with a broken gauge symmetry does not have the usual Goldstone mode (gapless excitations associated with the broken continuous symmetry in the ground state). Instead, it has a longitudinal plasma mode with a finite energy excitation gap analogous the the finite mass of the vector meson. This effect is due to coupling of the order parameter of the broken symmetry to the electromagnetic field. This coupling of the electromagnetic field to the order parameter is an effective way of describing the long range electromagnetic interactions between electrons. In this way, the photon in superconductivity acquires a mass. The Meissner effect follows naturally: the electromagnetic fields from sources outside a bulk superconductor cannot propagate far into the bulk since the propagation of the electromagnetic fields in the superconductor is carried out by massive photons and is thus short-ranged, just as the propagation of nuclear force fields by massive pions (in Yukawa's model) is short-ranged. Therefore, the

Higgs mechanism can provide a very simple understanding of the electromagnetic properties of superconductivity.

2.2 The Higgs mechanism in field theory

In this section, we will briefly review the Higgs mechanism in quantum field theory. Following Coleman[4], we will discuss the Higgs phenomenon in classical field theory (while using quantum field theory terms) since the basic features of the phenomenon are still the same when the field is quantized.

We start out with a model Lagrangian density for a complex scalar field with U(1) symmetry*,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \psi^*)(\partial^\mu \psi) - U(|\psi|) \; . \tag{2.1}$$

If one writes $\psi = |\psi| e^{i\phi}$, then the fact that Lagrangian is independent of ϕ indicates the U(1) symmetry, i.e., the Lagrangian is invariant under the transformation $\psi \to \psi e^{i\theta}$. The corresponding energy density is,

$$\mathcal{H} = \frac{1}{2}|\partial_0 \psi|^2 + \frac{1}{2}|\nabla \psi|^2 + U(|\psi|) \; . \tag{2.2}$$

From equation (2.2), it is easily seen that the vacuum is a state with $\psi = \psi_0 =$ constant, and $U(|\psi|)\Big|_{\text{min.}} = U(|\psi_0|)$. If $\psi_0 = |\psi_0| e^{i\phi_0} \neq 0$, then the vacuum state has broken the U(1) symmetry by choosing a fixed value of ϕ_0.

If one writes the Lagrangian density in terms of $|\psi|$ and ϕ,

$$\mathcal{L} = \frac{1}{2}\Big[\, \partial_\mu |\psi| \partial^\mu |\psi| + |\psi|^2 \partial_\mu \phi \partial^\mu \phi \Big] - U(|\psi|) \; , \tag{2.3}$$

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* Notation : the signature of the metric tensor is $(+ - - -)$; $\partial_\mu = \partial / \partial x^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$; summation over repeated indices is always implied.

one immediately recognizes that the particle represented by ϕ is massless (the Goldstone boson) since there is no mass term like $m^2\phi^2$ in the Lagrangian density. On the other hand, the mass for the particle represented by $|\psi|$ (or $\psi' \equiv |\psi| - |\psi_0|$) is proportional to $U''(|\psi_0|)$ and can be taken to be finite in our model.

To eliminate the Goldstone boson, one can enlarge the U(1) symmetry to a local U(1) symmetry. The new Lagrangian should be invariant under the transformation $\psi \rightarrow \psi e^{i\theta(x^\mu)}$. For an infinitesimal transformation, $\delta\partial_\mu\psi = i\partial_\mu\psi\delta\theta + i\psi\delta\partial_\mu\theta$, the second term indicates that the original Lagrangian (2.1) is not invariant under a local gauge transformation. But if one couples to ψ a gauge field A_μ that transforms according to

$$\delta A_\mu = -\frac{1}{e}\partial_\mu(\delta\theta) \tag{2.4}$$

in the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi^* + ieA_\mu\psi^*)(\partial^\mu\psi + ieA^\mu\psi) - U(|\psi|) \ , \tag{2.5}$$

one obtains a Lagrangian that is gauge invariant. The constant e is defined as the strength of the coupling. Furthermore, to make the gauge field a true dynamical variable, one can add the simplest gauge-invariant dynamical term $F_{\mu\nu}^2$ to the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\psi^* + ieA_\mu\psi^*)(\partial^\mu\psi + ieA^\mu\psi) - U(|\psi|) \ , \tag{2.6}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This is just the usual Lagrangian density of minimally-coupled electromagnetism with the usual physical interpretations of charged bosons (ψ field) and massless photons (A_μ field).

We can again write the Lagrangian in terms of ψ' and ϕ,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\Big[(\partial_\mu\psi')^2 + (\psi' + |\psi_0|)^2(\partial_\mu\phi + eA_\mu)^2\Big] - U(\psi' + |\psi_0|) \ . \tag{2.7}$$

Since $\partial_\mu \phi$ appears only in the form $\partial_\mu \phi + eA_\mu$, one can change variables,

$$a_\mu = A_\mu + e^{-1}\partial_\mu \phi \ , \tag{2.8}$$

then we have,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + \frac{e^2}{2}(\psi' + |\psi_0|)^2 a_\mu^2 + \frac{1}{2}(\partial_\mu \psi')^2 - U(\psi' + |\psi_0|) \ . \tag{2.9}$$

Now we can readily recognize that the zero mass ϕ field combines with the A_μ field to give a massive vector meson field a_μ with mass given by $e^2|\psi_0|^2$. The Goldstone boson has disappeared. It is eaten by the photon and the latter becomes heavy.

2.3 The Goldstone mode in a neutral superfluid

The ideas of the Goldstone boson and the Higgs phenomenon play important roles in condensed matter physics too. One only has to change the names: vacuum states in field theory correspond to ground states in condensed matter physics; the fields representing bose particles correspond to order parameters; and the mass corresponds to the excitation energy gap. Thus, a Goldstone boson in field theory is called a Goldstone mode (a gapless excitation above the ground state due to a broken continuous symmetry) in condensed matter physics. Phonons in crystals, spin waves in ferromagnets and sound-wave-like collective modes in superfluids are all Goldstone modes.

In this section, we will show that a Bose-condensed neutral superfluid (e.g. Helium II) has a ground state with a broken gauge symmetry and, as a result of the broken gauge symmetry, a Goldstone mode (phonon mode) exists. We will derive its dispersion relation. Finally, we will write down a Lagrangian density that gives the same physics as above so that we can draw some analogies with the

Goldstone boson in field theory and build up a basis upon which we can develop a Lagrangian formulation for a charged superfluid.

Penrose and Onsager[5] have shown that a superfluid of Bose particles (like Helium II) can be defined as a state in which the density matrix $\rho(\mathbf{r},\mathbf{r}')$ can factorize:

$$\rho(\mathbf{r},\mathbf{r}') \equiv \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle = \psi^*(\mathbf{r})\psi(\mathbf{r}') + \rho'(\mathbf{r},\mathbf{r}') \ , \tag{2.10}$$

with $\rho'(\mathbf{r},\mathbf{r}') \to 0$ as $|\mathbf{r}-\mathbf{r}'| \to \infty$. Following the work of Beliaev[6] who showed that the factorization can be extended to the time-dependent Green's function, Anderson[7] further emphasized that one can take the definition of a superfluid as a fluid in which the particle field operator $\hat{\psi}$ has a macroscopic mean value (superfluid order parameter),

$$\langle \hat{\psi}(\mathbf{r},t)\rangle = \psi(\mathbf{r},t) = |\psi| e^{i\phi} \ . \tag{2.11}$$

He then pointed out that one can deal with $|\psi|$ and ϕ as though they are thermodynamic variables of the system and one can also define coarse-grained averages of them just as one deals with usual thermodynamic variables. The ground state is one in which $\psi(\mathbf{r},t)$ is constant throughout the whole system. Furthermore, unlike $|\psi|$, ϕ is also a dynamical variable with the number operator N as its conjugate,

$$[\phi, N] = i \ , \tag{2.12}$$

with the operator equivalences

$$N = -i\partial/\partial\phi, \quad \phi = i\partial/\partial N \ . \tag{2.13}$$

Thus we have,

$$i\hbar\dot{N} = [\mathcal{H}, N] = i\frac{\partial\mathcal{H}}{\partial\phi} \ , \tag{2.14}$$

$$i\hbar\dot{\phi} = [\mathcal{H}, \phi] = -i\frac{\partial\mathcal{H}}{\partial N} \ . \tag{2.15}$$

One can take N and ϕ as the operators for the whole system as well as for a piece of the system while $\mathcal{H}$ is always interpreted as the Hamiltonian for the whole system. From (2.14), the number conservation for the whole system ($\dot{N} = 0$) implies that the system has a U(1) symmetry (global gauge symmetry) $\partial\mathcal{H}/\partial\phi = 0$. The fact that the ground state of superfluid has constant ϕ means that the gauge symmetry is broken. It also means that for a small static variation of the ϕ field, the energy density should be a quadratic function of $\nabla\phi$ as

$$E = n_s \frac{\hbar^2}{2m} (\nabla\phi)^2 \ , \tag{2.16}$$

where n_s is a arbitrary parameter and should be the number density of particles in the system at absolute zero temperature in a perfectly homogeneous system. Combining (2.14) and (2.16) (while treating N and ϕ as operators for small pieces in the system), one finds that the particle flow current is,

$$\mathbf{J_s} = n_s \frac{\hbar}{m} \nabla\phi \ . \tag{2.17}$$

Therefore, we can treat $\mathbf{V_s} \equiv \frac{\hbar}{m}\nabla\phi$ as the superfluid velocity for a homogeneous system (with very dilute or no impurities) at zero temperature.

Equation (2.15) will then give an acceleration equation of the superfluid. Taking the gradient of (2.15) gives,

$$\hbar \frac{\partial}{\partial t} (\nabla\phi) = -\nabla\mu \ , \tag{2.18}$$

where μ is the chemical potential. In terms of $\mathbf{V_s}$, one has,

$$m \frac{\partial \mathbf{V_s}}{\partial \mathbf{t}} = -\nabla\mu \ . \tag{2.19}$$

As one might have observed, equation (2.19) is not Galilean invariant. This is because we have chosen a reference frame in which the ground state has $\mathbf{V_{s0}} = 0$

and we can thus ignore the $\mathbf{V_s}^2$ term in the equation for small variations of the ϕ field.

Now we are ready to study the excitations associated with small variations of the ϕ field. Differentiating (2.19) with respect to t gives,

$$\begin{aligned} m\frac{\partial^2 \delta\mathbf{V}_s}{\partial t^2} &= -\frac{\partial}{\partial t}(\nabla\delta\mu) \\ &= -\frac{\partial\mu}{\partial\rho}\nabla\frac{\partial\delta\rho}{\partial t} \\ &= \frac{\partial\mu}{\partial\rho}\rho\nabla(\nabla\cdot\delta\mathbf{V}_s) \\ &= \frac{\partial P}{\partial\rho}\nabla^2\delta\mathbf{V}_s \ , \end{aligned} \tag{2.20}$$

where ρ is the particle number density. Through the derivations we have used the relations

$$\nabla\times\mathbf{V}_s \propto \nabla\times\nabla\phi = 0 \ , \tag{2.21}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}_s) = 0 \ . \tag{2.22}$$

We can then easily obtain the dispersion relation from (2.20),

$$\omega = c_p k \ , \tag{2.23}$$

where $c_p \equiv \frac{1}{m}\frac{\partial P}{\partial\rho}$. This is just the Goldstone mode related to a periodic variation of the ϕ field in space and time. It is the phonon mode (for small k) first predicted by Landau for Helium II.

We can also describe the dynamics of the ϕ field with a Lagrangian formulation while understanding that we use the Lagrangian in a classical field theory sense : the field equations (Euler-Lagrange equations) are all the physical content we can get from them. We can easily construct

$$\mathcal{L} = n_s\frac{\hbar^2}{m}\Big[\frac{1}{c_p^2}(\frac{\partial\phi}{\partial t})^2 - (\nabla\phi)^2\Big] \tag{2.24}$$

to describe the ϕ field, where n_s is a constant representing the equilibrium number density and ϕ will be able to describe density fluctuations. The Lagrangian (22) will give the correct field equation for ϕ, $\frac{\partial^2\phi}{\partial t^2} - c_p^2\nabla^2\phi = 0$, and thus the right dispersion relation (2.23). To compare the Lagrangian (2.24) and the Lagrangian (2.3) in Goldstone's complex scalar field model, we see that the degree of freedom represented by ϕ in (2.24) is closely analogous to that represented by ϕ in (3). Both have U(1) symmetry and lead to an excitation mode without an excitation gap. One minor difference is that the Lagrangian (2.24) is not Lorentz covariant simply because c_p, which gives the correct dynamics of the ϕ field in (2.24), is not equal to the speed of light c. This will only affect the dispersion relations, especially (as we shall soon see) when the ϕ field is coupled to a vector gauge field.

Before finishing this section, we want to point out that it is much easier to construct a Lagrangian formulation for a pure system at zero temperature than for a system at finite temperature due to the absence of dissipation in the zero temperature case. It is especially difficult to extend this formalism to study the dynamic properties of the order parameter field at finite temperature, since a theory with a Lagrangian formulation necessarily implies the time reversal symmetry which a superfluid at finite temperature does not have due to dissipation. On the other hand, it is relatively easy (and thus conventional) to study dissipative systems by directly setting up the equations of motion of the order parameters. The extension of Lagrangian formulation for a dissipative system has been studied at length by various people (see e.g. Caldeira and Leggett[8] for the study of quantum tunneling in dissipative systems). For our purpose here, we are more interested in obtaining dispersion relations for excitation modes than in discussing the propagation of them. Thus we will restrict our discussions to the non-dissipative case which should be a good approximation for clean systems at very low temperature.

2.4 Higgs mechanism in superconductivity

We have just shown that a neutral superfluid has a broken gauge symmetry and a Goldstone mode in close analogy with the broken U(1) symmetry and the Goldstone boson in Goldstone's complex scalar field model. In this section, we will show that the extension of the theory for neutral superfluids to a theory of superconductors produces an analogy to the Higgs mechanism; the phonon mode of the neutral superfluid is lifted to a plasma mode, photons become massive, and the Meissner effect follows as a consequence.

We can readily extend the previous discussion to superconductors, i.e., charged superfluids. Gor'kov[9] showed that for superconductivity the order parameter should be defined as the mean value of the electron pair field operator $\psi(\mathbf{r},t) \equiv \langle \hat{\psi}_{\downarrow}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r})\rangle$. Thus equations (2.12) — (2.15) should still hold if one takes an electron pair as a superfluid "particle". On the other hand, equation (2.16) cannot be right since $\nabla\phi$ does not have physical meaning: a local gauge transformation can change $\nabla\phi$ without producing any physical change. One can easily cure this by writing

$$E = n_s \frac{\hbar^2}{2m^*}(\nabla\phi + \frac{e^*}{\hbar c}\mathbf{A})^2 \ , \tag{2.25}$$

where $e^* = 2|e|$, $m^* = 2m_e$, and n_s is the equilibrium density of electron pairs. Consequently, one finds the electron pair velocity and charge current:

$$\mathbf{V_s} = \frac{\hbar}{m^*}(\nabla\phi + \frac{e^*}{\hbar c}\mathbf{A}) \ , \tag{2.26}$$

$$\mathbf{J_s} = n_s(-e^*)\mathbf{V_s} \ . \tag{2.27}$$

With $\langle\frac{\partial\mathcal{H}}{\partial N}\rangle = \mu - e^*\Phi$, where Φ is the electrostatic potential, the acceleration equation (2.19) becomes

$$\begin{aligned} m^*\frac{\partial\mathbf{V_s}}{\partial t} &= -\nabla(\mu - e^*\Phi) + \frac{e^*}{c}\frac{\partial\mathbf{A}}{\partial t} \\ &= -\nabla\mu - e^*\mathbf{E} \end{aligned} \tag{2.28}$$

for charged superfluids. The extra electric field term $(-e^*)\mathbf{E}$ will provide a finite restoring force even for very large wavelengths of variation of the ϕ field and give an excitation gap for the dispersion relation ($\mathbf{B} = 0$ for longitudinal mode):

$$\begin{aligned} m^* \frac{\partial^2 \delta \mathbf{V}_s}{\partial t^2} &= -\frac{\partial}{\partial t} \nabla \mu - e^* \frac{\partial \mathbf{E}}{\partial t} \\ &= \frac{\partial P}{\partial \rho} \nabla^2 \delta \mathbf{V}_s - 4\pi n_s e^{*2} \delta \mathbf{V}_s \ . \end{aligned} \tag{2.29}$$

The dispersion relation for this longitudinal mode (longitudinal photon) is,

$$\omega^2 = \omega_p^2 + c_p^2 k^2 \ , \tag{2.30}$$

where $\omega_p \equiv (4\pi n_s e^{*2}/m^*)^{1/2}$ is the plasma frequency.

We can also obtain London's equation from (2.26) and (2.27),

$$\nabla \times \mathbf{J}_s = -\frac{n_s e^{*2}}{m^* c} \mathbf{B} \ . \tag{2.31}$$

Combining this with the Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E} + \dfrac{1}{c} \dfrac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \times \mathbf{B} - \dfrac{1}{c} \dfrac{\partial \mathbf{E}}{\partial t} = \dfrac{4\pi}{c} \mathbf{j} \end{cases} \tag{2.32}$$

will give dispersion relation for the transverse modes (transverse photons),

$$\omega^2 = \omega_p^2 + c^2 k^2 \ . \tag{2.33}$$

For the static situation ($\omega \to 0$), $\mathbf{E} \to 0$, $k \to i\frac{\omega_p}{c}$, we get the Meissner effect with the London penetration depth $\lambda_L = \frac{c}{\omega_p}$.

We have observed that by attaching electric charge to superfluid particles, one has to formulate a theory with local gauge symmetry. After doing this, one finds that the original Goldstone mode due to the global gauge symmetry (equation (2.23)) is lifted to have a gap (equation (2.30)). The additional transverse mode

(2.33) can be thought of as the modification of the transverse electromagnetic field propagation due to the presence of the charged superfluid.

One can see a closer analogy between this phenomenon and the Higgs mechanism in field theory by formulating a Lagrangian description with local gauge symmetry extended from the global gauge symmetry U(1) of equation (2.24):

$$\mathcal{L} = n_s \frac{\hbar^2}{2m^*} \left[\frac{1}{c_p^2} \left(\frac{\partial \phi}{\partial t} - \frac{e^*}{\hbar} \Phi \right)^2 - \left(\nabla \phi + \frac{e^*}{\hbar c} \mathbf{A} \right)^2 \right] - \frac{1}{16\pi} F_{\mu\nu}^2 , \tag{2.34}$$

where the $F_{\mu\nu}^2$ term provides the dynamics for the gauge field $A^\mu = (\Phi, \mathbf{A})$. This general form for the Lagrangian agrees with the time-dependent Ginsburg-Landau equation for superconductivity at zero temperature (see Appendix 2A). We can briefly show that this Lagrangian indeed gives results consistent with what we obtained before.

We can conveniently combine the ϕ and A^μ degrees of freedom into

$$a^\mu = A^\mu - \frac{e^*}{\hbar c} \partial^\mu \phi . \tag{2.35}$$

Then the Lagrangian in terms of the a^μ fields is,

$$\mathcal{L} = \frac{\omega_p^2}{8\pi c^2} \left(\frac{c^2}{c_p^2} a_0^2 - \mathbf{a}^2 \right) - \frac{1}{16\pi} (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 . \tag{2.36}$$

The Lagrangian (2.36) has the same structure with the Lagrangian (2.9) for the a_μ fields and thus one can readily conclude that the Goldstone mode in a neutral superfluid has combined with the massless photons (the two transverse electromagnetic field degrees of freedom) to become massive photons. The Goldstone mode is lifted to become the longitudinal photon (plasma mode) and the two transverse photons also acquire the same mass ($\hbar\omega_p$). But (2.36) is not Lorentz covariant, as indicated by the fact that the Lagrangian distinguishes between the time and spatial components of the a_μ field. We can briefly show that this leads to different dispersion relations for the transverse and longitudinal photons.

The Euler-Lagrange equations for the a_μ fields are,

$$\frac{\partial \mathcal{L}}{\partial c_\mu} - \frac{1}{4\pi}\partial_\nu(\partial^\mu c^\nu - \partial^\nu c^\mu) = 0 \ . \tag{2.37}$$

Operating with ∂_μ upon (2.37) gives,

$$\frac{1}{c}\frac{\partial a_0}{\partial t} + \frac{c_p^2}{c^2}\nabla \cdot \mathrm{a} = 0 \ . \tag{2.38}$$

The explicit form of (2.37) separates out into time and space parts as,

$$\begin{cases} \partial_\mu \partial^\mu a_0 - \dfrac{\omega_p^2}{c_p^2} a_0 + \dfrac{1}{c}\dfrac{\partial}{\partial t}(\partial_\mu a^\mu) = 0 \ , \\ \partial_\mu \partial^\mu \mathrm{a} - \dfrac{\omega_p^2}{c^2}\mathrm{a} + \nabla(\partial_\mu a^\mu) = 0 \ . \end{cases} \tag{2.39}$$

Straightforward calculations by combining (2.38) and (2.39) can then show that the dispersion relations are just as in (2.30) and (2.33), and that the difference between longitudinal and transverse modes is due to this separation of space and time component behaviors of the field. Ultimately, the difference is due to the fact that the Lagrangian is not covariant, since a Lorentz invariant theory cannot separate transverse modes from a longitudinal mode and thus has to give the same dispersion relations for them.

The main result is that, except for the minor difference caused by the problem of Lorentz invariance, the Higgs mechanism in field theory and the photon's acquisition of mass in superconductivity can indeed be formulated in the same manner.

2.5 Discussions and conclusions

The discussions for superconductivity in the last section all seem very simple. So it is necessary here for us to comment about their validity and about the approximations we have made throughout the discussions.

The method we used above to obtain collective excitation modes of superconductivity can be thought of as a mean field approximation. Strictly speaking, one would have to consider the residual interactions between all electron pairs and solve the equations of motion for this many-body system to obtain exact results, where residual interactions (including the long range part of the Coulomb interaction) represent those interactions between electrons that have not been taken into account during the pairing of electrons. This problem is very similar to the electron gas problem. In fact, our dispersion relations (2.30) and (2.33) are very similar to those of plasmons in the electron gas. For the electron gas problem, Bohm and Pines[10] showed (using the Random Phase Approximation) that for low momentum modes (with wavelength larger than the neighboring electron separation), the many-body electron-electron interactions can be well described by letting each electron interact with the well defined collective field modes (plasma modes). These plasma modes (with electrons moving coherently) have transverse dispersion like (2.33) and a longitudinal dispersion relation $\omega = \omega_p$. The random motion of electrons (as opposed to the coherent motion with the plasma field) gives another effective restoring term $\langle v^2 \rangle k^2$ for the longitudinal mode through the interactions between the electrons and the collective field. This effect of the random motion of electrons (with a restoring term $\langle v^2 \rangle k^2$) is of the same physical nature as the effect of pressure (with a restoring term $c_p^2 k^2$ in (2.30)) in our fluid model. Classically, it simply means that when we reduce a Boltzmann description with both real space and momentum space distributions of particles into a fluid model description (the one we have here) with only a real space distribution of particles, we can effectively represent the effects of velocity randomness by the concepts of pressure P and the force ∇P associated with any non-uniform spatial distribution. Thus for our superconductivity problem, one would expect (2.30) and (2.33) to be reasonable for weak electromagnetic fields and for wavelengths larger than the pair correlation length. The validity of the acceleration equation (2.28) requires that the total effective electromagnetic field be weak, and the validity of the definition of the course-grained average of the ϕ field requires that our discussion be restricted to phenomena with length scales much larger than the electron pair correlation length.

It is also interesting to compare our results with results that have been already established through more rigorous calculations. Historically, the search for collective modes in superconductivity is related to the fact that the original BCS theory does not give a gauge invariant description of the electromagnetic response of a superconductor. Anderson[11] made the first attempt to obtain a gauge invariant theory of superconductivity by going beyond the Hartree-Fock-like approximation in the original BCS theory. He used the Random Phase Approximation to set up a time dependent Hartree-Fock-like approximation scheme. He was able to prove that in addition to the single particle excitation spectrum correctly given by BCS theory, there also exist longitudinal collective excitations. These longitudinal waves have a velocity $v_F\{\frac{1}{3}[1-4N(0)|V|]\}^{1/2}$ for a neutral Fermi superfluid and become the plasma oscillations in the charged case. Following Anderson's work and the improvement by Rickayzen,[12] Ambegaokar and Kadanoff[13] gave a clear calculation of the gauge invariant response of superconductors to weak electromagnetic fields and also obtained results similar to Anderson's. They showed that in the long wavelength limit the collective mode can be described as a state in which the phase ϕ of the order parameter $\psi = |\psi|e^{i\phi}$ varies periodically in space and time while $|\psi|$ remains fixed. Our approach gives the same description. Here we can also use our Lagrangian formulation (2.34) to derive the electromagnetic response of superconductors and show that they agree with the results given by Ambegaokar and Kadanoff.

In equation (2.34), from $\frac{\partial \mathcal{L}}{\partial \phi} = 0$, one has $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$. Since local gauge invariance is always related to local charge conservation, we expect the four current j^μ to be proportional to $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$, and indeed one has,

$$\begin{aligned} j^\mu &= \frac{e^*}{\hbar} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\ &= (-e^*) n_s \frac{\hbar}{m^*} \left(-\frac{c}{c_p^2} \left(\frac{\partial \phi}{\partial t} - \frac{e^*}{\hbar} \Phi \right), \left(\nabla \phi + \frac{e^*}{\hbar c} \mathbf{A} \right) \right) \end{aligned} \tag{2.40}$$

and

$$\partial_\mu j^\mu = 0 \, . \tag{2.41}$$

For long wavelength effects, we can Fourier transform equation (2.40) and (2.41) and solve for $\delta\phi(\omega, \mathbf{k}) = \phi(\omega, \mathbf{k}) - \phi_0$. We have,

$$\delta\phi(\omega, \mathbf{k}) = i \frac{e^*}{\hbar c} \frac{c\omega \Phi(\omega, \mathbf{k}) - c_p^2 \mathbf{k} \cdot \mathbf{A}(\omega, \mathbf{k})}{\omega^2 - c_p^2 k^2} \, . \tag{2.42}$$

Substituting (2.42) into (2.40) then gives the correct gauge invariant electromagnetic response of superconductors,

$$\begin{cases} \langle \rho(\omega, \mathbf{k}) \rangle = \dfrac{n_s e^{*2}}{m^*} \left[\dfrac{c k^2 \Phi(\omega, \mathbf{k}) - \omega \mathbf{k} \cdot \mathbf{A}(\omega, \mathbf{k})}{\omega^2 - c_p^2 k^2} \right] \, , \\ \langle \mathbf{j}(\omega, \mathbf{k}) \rangle = -\dfrac{n_s e^{*2}}{m^* c} \left[\mathbf{A}(\omega, \mathbf{k}) + \dfrac{c_p^2 \mathbf{k} [\mathbf{k} \cdot \mathbf{A}(\omega, \mathbf{k})] - c\omega \mathbf{k} \Phi(\omega, \mathbf{k})}{\omega^2 - c_p^2 k^2} \right] \, . \end{cases} \tag{2.43}$$

These agree with the results obtained by Ambegaokar and Kadanoff.

In conclusion, we have shown that the theory of superconductivity develops from the theory of neutral superfluids in a way closely analogous to the Higgs mechanism in quantum field theory. The Lagrangian formulation we give provides a mathematical structure upon which analogies with Higgs mechanism can be drawn explicitly. Photons in superconductivity thus acquire a mass and the Meissner effect can be understood naturally in a gauge-invariant way. One still needs to do many-body calculations[11,12,13] to understand the microscopic origin of the broken symmetry and the temperature and material dependent properties of superconductors. But with broken symmetry as the only basic starting point, one is able to understand both the basic features and the electromagnetic response of superconductors.

Appendix 2A

Connection with the Ginzburg-Landau Theory

We will now very briefly discuss the connection between our Lagrangian formulation and the Ginsburg-Landau theory.

One can expand the Lagrangian for the ϕ order parameter (2.34) into a Lagrangian for $\psi = |\psi| e^{i\phi}$ with a local gauge symmetry. A natural way to do so is to write

$$\mathcal{L} = n^* \frac{\hbar^2}{2m^*} \left[\frac{1}{c_p^2} \left| \left(\frac{1}{i} \frac{\partial}{\partial t} - \frac{e^*}{\hbar} \Phi \right) \psi \right|^2 - \left| \left(\frac{1}{i} \nabla + \frac{e^*}{\hbar c} \mathbf{A} \right) \psi \right|^2 \right] - U(|\psi|) - \frac{1}{16\pi} F_{\mu\nu}^2 . \tag{2A1}$$

It is easy to check that upon defining $n^* |\psi|^2 = n_s$, one would obtain the same Lagrangian for the ϕ field as that of (2.34). For a time-independent description, dropping the $\left(\frac{1}{i} \frac{\partial}{\partial t} - \frac{e^*}{\hbar} \Phi \right) \psi$ term, one finds that $\mathcal{L}$ is the same as the original Ginsburg-Landau free energy except for an overall minus sign and for the detailed form of $U(|\psi|)$ in the Ginsburg-Landau free energy. Furthermore, from the work of Abrahams and Tsuneto,[14] we know that for a pure superconductor (or a superconductor with very dilute impurities) at zero temperature the time-dependent generalization of the Ginsburg-Landau theory for small space and time variations of ψ can be formulated with a Lagrangian density

$$\begin{aligned} \mathcal{L} = \frac{1}{2} N(0) \Bigg[& \frac{(|\psi|^2 - |\psi_0|^2)^2}{|\psi_0|^2} + \frac{v_F^2}{3} \left| \left(\frac{\hbar}{i} \nabla + \frac{e^*}{c} \mathbf{A} \right) \frac{\psi}{\psi_0} \right|^2 \\ & - \left| \left(\frac{\hbar}{i} \frac{\partial}{\partial t} - e^* \Phi \right) \frac{\psi}{\psi_0} \right|^2 \Bigg] + \frac{H^2 - E^2}{8\pi} , \end{aligned} \tag{2A2}$$

which agrees with our general form of (2.42). Comparing (2.42) and (2.43) one has,

$$c_p^2 = \frac{v_F^2}{3} = \frac{n_s}{m^* N(0)} .$$

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Chapter 3

Meissner Effects in Solids: Expulsion of Twist Distortions

Many physical systems display certain rigidity when a symmetry is broken: A superconductor expels magnetic field when gauge symmetry is broken and the superconducting order parameter is nonzero; a smectic liquid crystal expels twist and bend distortions under appropriate boundary conditions when the translational symmetry of a nematic liquid crystal is broken and the nematic liquid crystal becomes a smectic liquid crystal. In this chapter, we will show that a cubic-liquid-crystal phase — an intermediate phase between liquid and solid, with a translational symmetry but a broken rotational symmetry for the nearest neighbor bonds — will start to expel certain bond rotational distortions such as twist when the translational symmetry is also broken and the cubic-liquid-crystal becomes a solid. This is a rigidity associated with the broken translational symmetry. We will then explain that this special rigidity against twist distortions is a result of the expulsion of any distortions in which the local average nearest neighbor bond direction is not parallel to the direction of lines connecting nearest peaks of local density.

We study this effect in the context of Landau's theory of phase transitions. Following Nelson and Toner[1], we first construct a free energy density valid for the cubic-liquid-crystal to solid phase transition. In the cubic-liquid-crystal phase, the rotational symmetry of the liquid has already been broken, and the system can be characterized by a local set of rotation angles $\vec{\theta}(\vec{r})$, measured from a preferred orthonormal triad. Density fluctuations leading to a crystal can be characterized by the Fourier transform of density at a discrete set of minimal reciprocal-lattice vectors $\{\vec{G}_\alpha\}$ with a fixed orientation relative to the preferred triad. The rotationally invariant free energy density describing the phase transition is,

$$\begin{aligned}\mathcal{F} =&\frac{1}{2}A\sum_{\alpha}\left|\vec{G}_\alpha \times (\nabla - i\vec{G}_\alpha \times \vec{\theta}\,)\rho_{\vec{G}_\alpha}\right|^2 \\ &+\frac{1}{2}B\sum_{\alpha}\left|\vec{G}_\alpha \cdot (\nabla - i\vec{G}_\alpha \times \vec{\theta}\,)\rho_{\vec{G}_\alpha}\right|^2 + \frac{1}{2}s\sum_{\alpha}\left|\rho_{\vec{G}_\alpha}\right|^2 \\ &+t \sum_{\substack{\alpha,\beta,\gamma \\ \vec{G}_\alpha+\vec{G}_\beta+\vec{G}_\gamma=0}} \rho_{\vec{G}_\alpha}\rho_{\vec{G}_\beta}\rho_{\vec{G}_\gamma} + O(\rho^4_{\vec{G}_\alpha}) + \frac{1}{2}K_a|\vec{\nabla}\times\vec{\theta}\,|^2 + \frac{1}{2}K_b(\vec{\nabla}\cdot\vec{\theta}\,)^2 \ ,\end{aligned} \tag{3.1}$$

where s is assumed to decrease linearly with decreasing temperature, and t is a constant. For sufficiently small s, taking into consideration the cubic ρ term, and if necessary, the quartic ρ terms, we can have

$$\langle\rho_{\vec{G}_\alpha}\rangle \neq 0\ . \tag{3.2}$$

A solid phase is therefore formed and A and B are then related to the elastic constants of the solid. The A and B energy terms represent the energy cost for the new solid phase to have non-ideal spacing and direction between nearest peaks of density maxima. The Frank constants K_a and K_b are stiffness intrinsic to the cubic-liquid-crystal phase.

In the cubic-liquid-crystal phase, with $\langle\rho_{\vec{G}_\alpha}\rangle = 0$, the twist distortion $(\vec{\nabla}\cdot\vec{\theta} \neq 0)$ is distributed uniformly throughout the system when the cubic-liquid-crystal is under external twist stress. As the system enters into the solid phase with a broken translational symmetry with $\langle\rho_{\vec{G}_\alpha}\rangle \neq 0$, the system starts to expel twist distortions, and external twist perturbations can only penetrate a finite length (penetration depth) into the system. We will show this effect explicitly using free energy (3.1) and making a few mathematical simplifications.

For mathematical convenience and without influencing the physical results of the argument, we make the following simplifications. First, we assume the set of minimal reciprocal lattice vectors $\{\vec{G}_\alpha\}$ form a simple cubic lattice. This in turn gives $t = 0$. Second, we assume that the transition into the solid phase is either second order or weekly first order, so that the density fluctuations can be well described by the density Fourier transform at the three minimal reciprocal vectors. Third, we let $B = A$ for mathematical convenience. The resulting simplified free energy density is

$$\begin{aligned} \mathcal{F} = & \frac{1}{2}A \sum_{\alpha=x,y,z} \left|(\nabla - i\vec{G}_\alpha \times \vec{\theta}\,)\rho_{\vec{G}_\alpha}\right|^2 + \frac{1}{2}s \sum_{\alpha} \left|\rho_{\vec{G}_\alpha}\right|^2 + O(\rho^4_{\vec{G}_\alpha}) \\ & + \frac{1}{2}K_a|\vec{\nabla} \times \vec{\theta}\,|^2 + \frac{1}{2}K_b(\vec{\nabla} \cdot \vec{\theta}\,)^2 . \end{aligned} \tag{3.3}$$

We write

$$\rho_{\vec{G}_\alpha} = |\rho_{\vec{G}_\alpha}| e^{i\phi_\alpha(\vec{r})} , \tag{3.4}$$

where $\phi_\alpha(\vec{r}) = \vec{G}_\alpha \cdot \vec{u}(\vec{r})$ and $u(\vec{r})$ represents the phonon displacement of the solid, i.e., the displacement of the points of maximum density away from their ideal positions. In the solid phase, to the lowest order of twist perturbations, we can assume $|\rho_{\vec{G}_\alpha}|$ to be constant at a given temperature. Then the Euler-Lagrange equations for the $\vec{\theta}$-perturbation and the density fluctuation phase perturbation $\phi_\alpha = \vec{G}_\alpha \cdot \vec{u}$ can be shown to be,

$$2A|\rho_{\vec{G}_\alpha}|^2 \vec{G}^2_\alpha(\vec{\theta} - \frac{1}{2}\nabla \times \vec{u}) + K_a \nabla \times (\vec{\nabla} \times \vec{\theta}\,) - K_b \nabla(\vec{\nabla} \cdot \vec{\theta}\,) = 0 , \tag{3.5}$$

and

$$\vec{\nabla} \times \vec{\theta} + \nabla^2 \vec{u} = 0 . \tag{3.6}$$

We can now easily understand the behavior of twist distortions in a solid with a weakly broken translational symmetry. Taking the divergence of equation (3.5), we have

$$\nabla^2(\vec{\nabla} \cdot \vec{\theta}\,) - \frac{1}{\lambda_t^2}(\vec{\nabla} \cdot \vec{\theta}\,) = 0 , \tag{3.7}$$

where

$$\lambda_b \equiv \sqrt{\frac{K_b}{2A|\rho_{\vec{G}_\alpha}|^2 \vec{G}_\alpha^2}} \tag{3.8}$$

is the penetration depth for twist perturbations. It can be shown quite generally[2] from equation (3.7) that twist deformations (with $\vec{\nabla}\cdot\vec{\theta} \neq 0$) exist only within length λ_b of the boundary of the solid and they decay exponentially (with decay length λ_b) into the bulk of the system. (For completeness, we have briefly summarized the proof in Appendix 3A.) Therefore the broken translational symmetry represented by order parameter $\rho_{\vec{G}_\alpha} \neq 0$ effectively expels the twist deformations from the bulk of the solid. This process of rotational rigidity due to the breaking of translational symmetry is indeed analogous to the Meissner effect in superconductors.

But the twist distortion is not the only physical quantity that is expelled from the bulk of the system when a cubic-liquid-crystal phase makes a transition into a solid phase. In fact, the expulsion of twist is a natural result of the expulsion of a large class of distortions which we will define and discuss in the following.

As we discussed earlier, the local orientation of the cubic-liquid-crystal is defined through the direction of the nearest neighbor bonds and these bonds tend to be parallel to a preferred orthonormal triad. As the system further breaks the translational symmetry, it forms non-uniform density with nonzero density Fourier components at vectors $\vec{G}_\alpha$ ($\alpha = x, y, z.$). The directions of lines connecting the nearest peaks of density maxima tend to be parallel to another preferred orthonormal triad with a fixed orientation relative to the preferred triad of the cubic-liquid-crystal nearest neighbor bonds. (For symmetry reasons, these two preferred triad should be the same.) Due to fluctuations, the local nearest neighbor bond directions in a cubic-liquid-crystal can be away from its preferred orthonormal triad and the difference can be represented by a local set of rotation angle $\vec{\theta}(\vec{r})$. In the same way, the directions of local lines connecting nearest peaks of density can be also away from its preferred triad and the angular difference can be easily shown to be $\frac{1}{2}\vec{\nabla} \times \vec{u}$ for small phonon displacements. Due to rotational invariance, the difference between the above two small rotations, i.e.,

$$\vec{\xi} \equiv \vec{\theta} - \frac{1}{2}\vec{\nabla} \times \vec{u} \,, \tag{3.9}$$

is the real physical quantity that describes by how much the local average nearest neighbor bond directions are not parallel to the directions of lines connecting the nearest peaks of density maxima. This quantity is expelled from the bulk of the system, it decays exponentially away from the surface, and its expulsion causes twist to be expelled from the bulk of the system.

We can easily prove the above statements by first establishing the following relations:

$$\vec{\nabla} \cdot \vec{\xi} = \vec{\nabla} \cdot \vec{\theta} \,, \tag{3.10}$$

and

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{\xi}) &= \vec{\nabla} \times (\vec{\nabla} \times \vec{\theta}) - \frac{1}{2}\vec{\nabla} \times [\vec{\nabla} \times (\vec{\nabla} \times \vec{u})] \\ &= \vec{\nabla} \times (\vec{\nabla} \times \vec{\theta}) + \frac{1}{2}\nabla^2 \vec{\nabla} \times \vec{u} \\ &= \frac{1}{2}\vec{\nabla} \times (\vec{\nabla} \times \vec{\theta}) \,, \end{aligned} \tag{3.11}$$

where we used equation (3.6) for the last step of derivation. Putting these two relations back into the Euler-Lagrange equation (3.5), we obtain an equation for $\vec{\xi}$ field:

$$\nabla^2 \vec{\xi} - (1 - \frac{\lambda_{\mathrm{b}}^2}{\lambda_{\mathrm{a}}^2})\vec{\nabla}(\vec{\nabla} \cdot \vec{\xi}) - \frac{1}{\lambda_{\mathrm{a}}^2}\vec{\xi} = 0 \,, \tag{3.12}$$

where

$$\lambda_{\mathrm{a}} \equiv \sqrt{\frac{K_a}{A|\rho_{\vec{G}_\alpha}|^2 \vec{G}_\alpha^2}} \,. \tag{3.13}$$

We can decompose $\vec{\xi}$ field into a curl-free part $\vec{\xi}_{\mathrm{d}}$ and a divergence-free part $\vec{\xi}_{\mathrm{c}}$. Then by choosing $\vec{\xi}_{\mathrm{d}}$ and $\vec{\xi}_{\mathrm{c}}$ appropriately[3], the $\vec{\xi}$ field equation (3.12) can be written into two equations:

$$\begin{cases} \nabla^2\vec{\xi}_{\mathrm{d}} - \dfrac{1}{\lambda_{\mathrm{b}}^2}\vec{\xi}_{\mathrm{d}} = 0 \; ; \\ \nabla^2\vec{\xi}_{\mathrm{c}} - \dfrac{1}{\lambda_{\mathrm{a}}^2}\vec{\xi}_{\mathrm{c}} = 0 \; . \end{cases} \tag{3.14}$$

We see that the curl-free part of $\vec{\xi}$ field $\vec{\xi}_{\mathrm{d}}$ decays into the bulk with penetration depth λ_{b}, while the divergence-free part of the $\vec{\xi}$ field $\vec{\xi}_{\mathrm{c}}$ decays into the bulk with penetration depth λ_{a}. The $\vec{\xi}$ field is therefore excluded from the bulk of the solid. In the bulk of the system, the nearest neighbor bond angle distortions $\vec{\theta}$ is then locked with phonon displacement via

$$\vec{\theta} = \frac{1}{2}\vec{\nabla} \times \vec{u} \; . \tag{3.15}$$

And this is why in the normal description of a solid, the phonon $\vec{u}$ field completely describes the distortions.

The twist distortions of the nearest neighbor bonds exist only due to a nonzero $\vec{\xi}$ field, and it decays with the same penetration depth as that of the decay of a curl-free $\vec{\xi}$ field. We should notice that this twist is not the same as the "twist distortion" of a solid caused by, e.g., twisting a solid rod. Once we can use phonon field $\vec{u}$ to completely describe a solid, the local rotation of bonds away from its preferred triad is $\frac{1}{2}\vec{\nabla} \times \vec{u}$ and it is always twist-free because $\vec{\nabla} \cdot \frac{1}{2}\vec{\nabla} \times \vec{u} = 0$.

In summary, we have shown that the breaking of translational symmetry in a system with broken rotational symmetry causes the system to expel certain rotational distortions such as twist. The resulting solid with broken translational and rotational symmetry is rigid against twist perturbations. This is analogous to the Meissner effect in superconductors. Furthermore, for the cubic-liquid-crystal to solid transition, the twist distortion is just one of the distortions that are expelled from the bulk of the system. In general, any distortions in which the nearest neighbor bonds are not parallel to the lines connecting the nearest peaks of density are expelled from the bulk of the solid.

Appendix 3A

General Proof of the Meissner Expulsion

In this appendix, we consider a physical system in which a physical quantity u satisfies the following equation:

$$\nabla^2 u - \beta^2 u = 0 \ . \tag{3A1}$$

We will prove that the solutions of this equation always show the property of decaying very rapidly toward the interior of the system with β^{-1} as the decay length. (See Ref. 2 for the origin of this proof.)

We start from Green's theorem

$$\iiint_V (u\nabla^2 v - v\nabla^2 u) dV = \iint_S \Big(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\Big) dS \ , \tag{3A2}$$

for any two functions u and v which, as well as their first derivatices, are finite and continuous with a region of volumn V. Here ν is the direction normal to the boundary S of the volumn V and pointing toward outside. If we suppose further that u and v satisfy equation (3A1), then the left-hand side of equation (3A2) disappears. Let us choose v to be

$$v = \frac{1}{|\vec{r} - \vec{r}_P|} e^{-\beta|\vec{r}-\vec{r}_P|} \ . \tag{3A3}$$

Then v satisfies equation (3A1) everywhere except at the point P, where it has a singularity. In order to apply equation (3A2), we have to exclude this point from the volumn V over which we integrate. We do this by cutting out a spherical region of a small radius ϵ around the point P. We further confine the column with another sphere surface S_1 of a larger radius R centered at P and located entirely in the interior of out system.

Apply the Green's theorem to u and the chosen v within the above chosen volumn V, and let $\epsilon \to 0$, we have,

$$4\pi u(\vec{r}_P) = \frac{e^{-\beta R}}{R^2} \iint_{S_1} \left\{ u(1+\beta R) + \frac{\partial u}{\partial \nu} R \right\} dS \; . \tag{3A4}$$

Since we can also choose

$$v = \frac{1}{|\vec{r} - \vec{r}_P|} e^{\beta |\vec{r} - \vec{r}_P|} \tag{3A5}$$

as a solution of equation (3A1) with the same singularity at P, we obtain another representation of $u(\vec{r}_P)$ by exchanging β with $-\beta$:

$$4\pi u(\vec{r}_P) = \frac{e^{\beta R}}{R^2} \iint_{S_1} \left\{ u(1-\beta R) + \frac{\partial u}{\partial \nu} R \right\} dS \; . \tag{3A6}$$

Multiplying (3A4) by $e^{\beta R}$ and (3A6) by $e^{-\beta R}$ and subtracting, we obtain

$$u(\vec{r}_P) = \frac{\beta R}{\sinh \beta R} \cdot \bar{u}_{S_1} \; , \tag{3A7}$$

where

$$\bar{u}_{S_1} \equiv \frac{1}{4\pi R^2} \iint_{S_1} u \, dS \tag{3A8}$$

is the mean value of u on the sphere of radius R around P. Equation (3A7) is true for all points P and for all spheres located entirely within the system. This result indicates that the u field always decays exponentially into the interior of the system with β^{-1} as the decay length. The u field can only penetrate β^{-1} into the bulk of the system.

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3. When decomposing $\vec{\xi}$ field into a curl-free part $\vec{\xi}_{\mathrm{d}}$ and a divergence-free part $\vec{\xi}_{\mathrm{c}}$, i.e., $\vec{\xi} = \vec{\xi}_{\mathrm{d}} + \vec{\xi}_{\mathrm{c}}$, we have some freedom of choice: The decomposition process is invariant under the following transformations:

$$\begin{cases} \vec{\xi}'_{\mathrm{d}} = \vec{\xi}_{\mathrm{d}} + \vec{\nabla}\varphi \ ; \\ \vec{\xi}'_{\mathrm{c}} = \vec{\xi}_{\mathrm{c}} - \vec{\nabla}\varphi \ , \end{cases} \tag{3A9}$$

where φ field satisfies $\nabla^2\varphi = 0$. It is then easy to show that with this freedom, we can decompose $\vec{\xi}$ field appropriately so that we can decompose the $\vec{\xi}$ field equation (3.12) into equations (3.14).

Chapter 4

Meissner Effects in Smectic Liquid Crystals

de Gennes, in 1972, suggested an intriguing analogy between superconductors and smectics A[1]. He showed that the free energy governing the nematic to smectic A liquid crystal transition is similar to that of the normal metal to superconductor transition. In particular, he argued that bend and twist distortions in smectics A play similar roles to that of magnetic field in a superconductor and that they are excluded from the bulk of the smectics A with a finite penetration depth. In the past few years, Renn and Lubensky[2] have discovered that this analogy can be pushed further. They predicted that smectic liquid crystals containing chiral molecules can be in a twist-grain-boundary (TGB) state, just as type II superconductors under an external magnetic field can be in the Abrikosov flux lattice state. The predicted TGB phase was later observed experimentally by Goodby *et al.* [3] and Srajer *et al.* [4]. In this chapter, we point out that bend and twist are ***not always*** expelled from smectics A. This result, studied by Durand[5] and Clark[6], was discussed in de Gennes' book[7]. While our arguments apparently do not disturb the type II superconductor analogies in smectic liquid crystals of Renn and Lubensky because they mostly worked in the region close to the upper critical twist field when the twist distortions are not expelled anyway, they point out the subtleties of the rigidity caused by the broken translational symmetry in smectics A.

[illegible]

cylindrical coordinates $\mathbf{r} = (r, \theta, z)$, the order parameter is of the form $\Delta(\mathbf{r}) = \exp(-i\theta)\Delta(r)$ with $\Delta(r)$ real. By taking advantage of the cylindrical symmetry, the quasiparticle amplitudes may be written as[6]

$$\begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix} = e^{ik_z z} e^{i\mu\theta} e^{-i\sigma_z\theta/2} \begin{pmatrix} g_n^+(r) \\ g_n^-(r) \end{pmatrix}. \tag{5.22}$$

(The σ's are the usual Pauli matrices.) The Bogoliubov equations then take the form

$$\begin{aligned} \sigma_z \frac{\hbar^2}{2m_{\|}} \left\{ -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \left[\mu - \sigma_z \left(\frac{1}{2} + \frac{eA_\theta r}{\hbar}\right)\right]^2 \frac{1}{r^2} - k_\rho^2 \right\} \hat{g}_n(r) \\ + \sigma_x \Delta(r) \hat{g}_n(r) = \epsilon_n \hat{g}_n(r), \end{aligned} \tag{5.23}$$

where $\hat{g}_n$ is a two component spinor. We have assumed different effective masses $m_{\|}$ in the plane and m_z along the z direction, and defined a radial wavenumber k_ρ by

$$\frac{\hbar^2 k_\rho^2}{2m_{\|}} = E_F - \frac{\hbar^2 k_z^2}{2m_z}. \tag{5.24}$$

Since NbSe$_2$ is an extreme type II superconductor, the vector potential in Eq. (5.23) may be ignored in subsequent calculations. In terms of these $\hat{g}_n$ spinors, the pair potential $\Delta(r)$ is to be determined self consistently from

$$\Delta(r) = U_0 \sum_n g_n^+(r) g_n^-(r) [1 - 2f(\epsilon_n)] \tag{5.25}$$

and the local density of states is

$$N(\omega, r) = \sum_n [|g_n^+(r)|^2 \delta(\omega - \epsilon_n) + |g_n^-(r)|^2 \delta(\omega + \epsilon_n)]. \tag{5.26}$$

In these last two equations the summation includes an integration over the scattering states.

The numerical procedure that Joel Shore used determines the eigenvalues by a "shooting" method. We first factor out the asymptotic oscillatory behavior of the wavefunctions $\hat{g}_n(r)$, and then solve the resulting differential equation for a fixed μ and k_z by integrating far from the vortex axis to the axis. The requirement that the wavefunctions be well-behaved at the axis determines the eigenvalues and corresponding eigenfunctions. For the pair potential, we choose the form

$$\Delta(r) = \Delta_0 \tanh(d\, r/\xi_{||}). \tag{5.27}$$

We initially guess a value for d, and after the wavefunctions are determined we can then solve for $\Delta(r)$ from Eq. (5.25) in order to obtain a better estimate of d, if we assume that the shape of $\Delta(r)$ is accurately determined by only the bound states. This allows us to obtain an approximately self-consistent solution.

We want to point out here that although we did not perform a fully self-consistent calculation that would require the inclusion of the scattering states (with $\epsilon > \Delta_0$), as mentioned by Gygi and Schluter[17], we believe the physical results we obtained here are correct for the following reason. The local density of states for energies less than Δ_0 can be correctly calculated without considering the scattering states once we are given a value of d. Full self-consistency means that we have to include all the bound states and the scattering states to self-consistently determine d. But our calculated local density of states turns out to be remarkably insensitive to the value of d. For example, even changing d by a factor 5 or 10 does not change the results much (and certainly does not change the qualitative behavior). So whether or not we have included the scattering states for a full self-consistent calculation does not matter much.

The numerical results are shown in Figs. 5.3–5.5. We have assumed an isotropic three-dimensional material ($m_{||} = m_z$) and chosen parameters appropriate to $NbSe_2$.[18] The results for the local density of states are also remarkably insensitive to the values of these parameters. The eigenvalues for small μ and k_z

agree to better than 1% with the results of Caroli *et al.*[6] (See their Eq. (10), valid in the limit $\mu << E_F/\Delta_0$ and small k_z.) Figure 5.4 shows the *local* density of states at $r = 0$ and $r = \xi_{||}$, while Figure 5.5 shows the thermally broadened local density of states (which is proportional to the experimentally measured differential conductance dI/dV) at a temperature of 1.9 K. The thermal broadening is due to the finite temperature of the STM tip and the $NbSe_2$ sample, and is determined by convoluting the results in Figure 5.4 with the absolute value of the derivative of the fermi function as in equation (5.3). Note that at $r = \xi_{||}$, the local density of states displays considerable structure, but that this structure is smeared into one large peak once we account for the broadening. We do find that at somewhat lower temperatures or further from the vortex center, a double peak (with a dip in the differential conductance at zero-bias) persists after thermal broadening (although at large distances it will be washed out by the thermal broadening of the scattering states into the gap region). This prediction was later verified by a new experiment by Hess *et al.*[12], and we have displayed their experimental measurement in Figure 5.6.

Our results for the differential conductance are in satisfactory agreement with the experimental results of Hess *et al.*, although the peaks we obtain are about 2.5 times higher and somewhat narrower than theirs. We believe this discrepancy has two sources: (1) Intrinsic effects, *e.g.*, pair-breaking and dirt effects in the sample. (Hess *et al.* do find some sample dependence in the height of the peaks[19].) Rough calculations by Klein[20] and Joel Shore[21] show that a little dirt goes a long way in broadening the local density of states. So even though $NbSe_2$ is in the clean limit, dirt could probably account for most of the broadening. (2) Probe resolution, *e.g.*, the spatial resolution of the tip, and uncertainty in the tip temperature due to lack of tip equilibration with the bath. We could imagine accounting for broadening

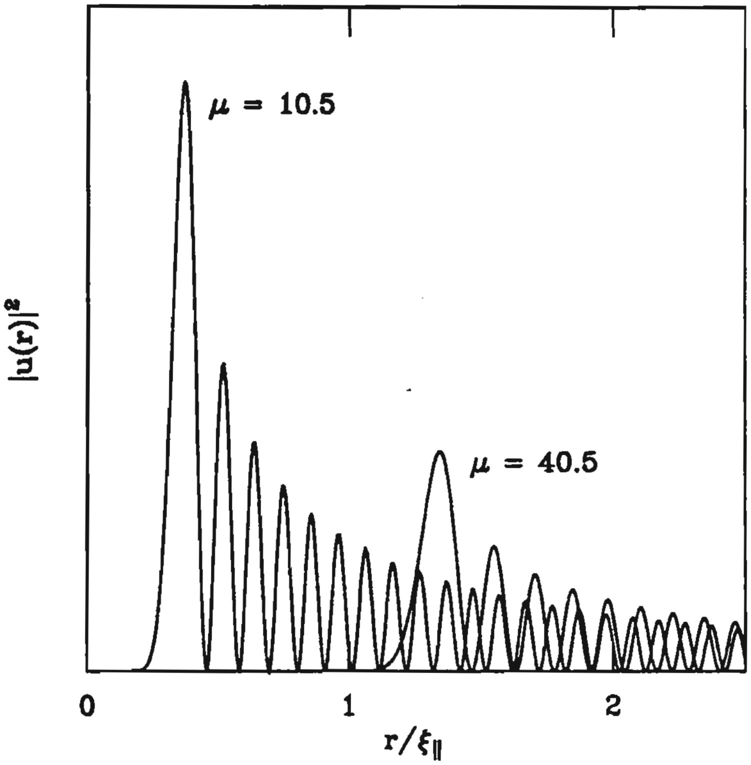

Figure 5.3 Quasiparticle amplitude $|u(r)|^2$ vs. the distance r from the vortex center for two different values of angular momentum μ, demonstrating that the first peak in the amplitude occurs further away from the center for larger μ (and hence larger energy). The parameters used here are $k_z = 0$, $E_F/\Delta_0 = 250$ and $d = 1.0$ (see Eq. (6)).

due to probe resolution by using an *effective* tip temperature of 3.5 K, which is obtained by fitting the energy width of the experimental peak at $r = 0$. Using this effective temperature would reduce the peaks in Figure 5.5 by about 40%.

Muzikar[22] pointed out that the assumption of the δ-function local attraction between electrons in the Bogoliubov equation formalism may also make the theoretical density of states peak sharper than it would be in a calculation that includes the Debye energy cutoff for electron-electron interactions. But a simple estimate

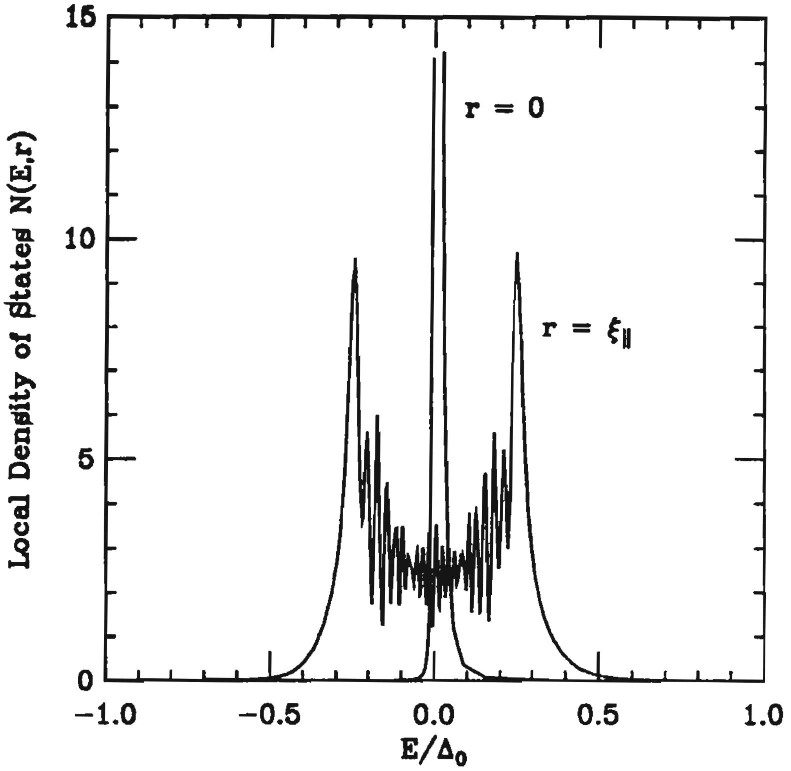

Figure 5.4 Calculated *local* electron density of states vs. energy, normalized by the fermi-level density of states in the normal metal, for $r = 0$ and $r = \xi_{\|}$. For $r = 0$, the peak is at $N(E,r) \approx 180$ (offscale). The parameters are $E_F/\Delta_0 = 250$ and $d = 1.0$. These parameter values yield a nearly self-consistent result for $\Delta(r)$ when $T = 1.9$ K.

can show that this assumption should not matter too much in the weak coupling limit. The BCS form of the phonon-mediated electron-electron attraction has the form of (with the Fermi energy set at zero)

$$V_{k,k'} = \begin{cases} -U_0, & \text{for } \epsilon_k, \epsilon_{k'} < \omega_D; \\ 0, & \text{otherwise.} \end{cases} \tag{5.28}$$

The width of the momentum space cutoff roughly introduces a real space finite range R of the electron-electron interactions:

$$R \sim \frac{1}{\Delta k} \sim \frac{\hbar v_F}{\hbar \omega_D} \sim \frac{\Delta_0}{\hbar \omega_D} \xi_0 \ll \xi_0 \; . \tag{5.29}$$

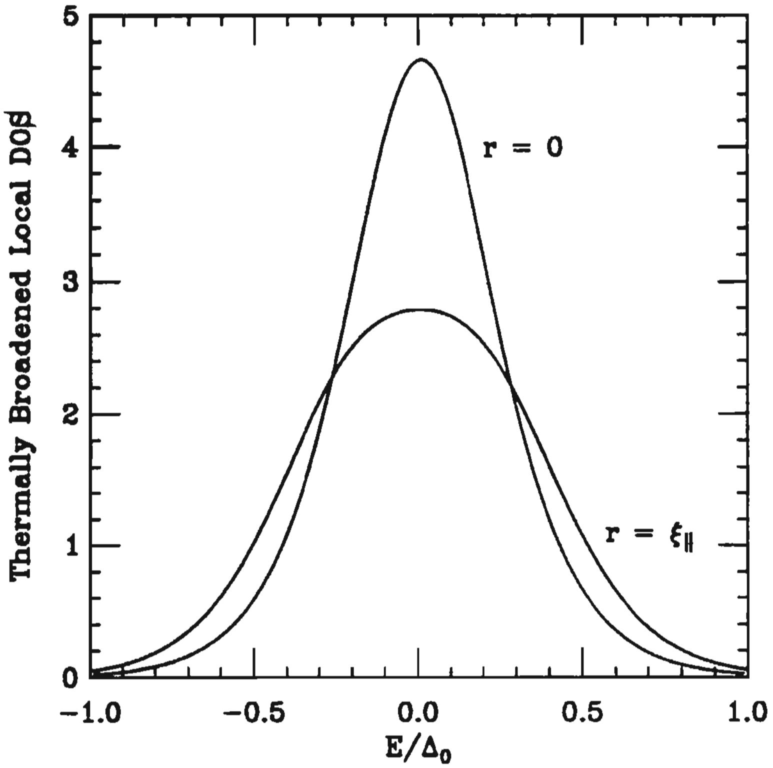

Figure 5.5 The same data shown in figure 3 after convolution with the derivative of the fermi function for a temperature of 1.9 K, in order to simulate the differential conductance that would be observed in the experiment. Note that in this and the previous figure, $\xi_{||}$ has been set so that the spatial width of the peak at zero bias approximately matches that seen in the experiment.

Since the interation range is much less than the superconductor correlation length in the weak coupling limit, it is safe to assume a δ-function local electron-electron interaction in the Bogoliubov equation formalism.

Furthermore, we can make a stronger statement about this assumption concerning the local density of states at the center of the vortex. Even with a non-local electron-electron interaction, we can still apply the generalized Hartree-Fock approximation to the problem and then diagonalize the new effective Hamiltonian

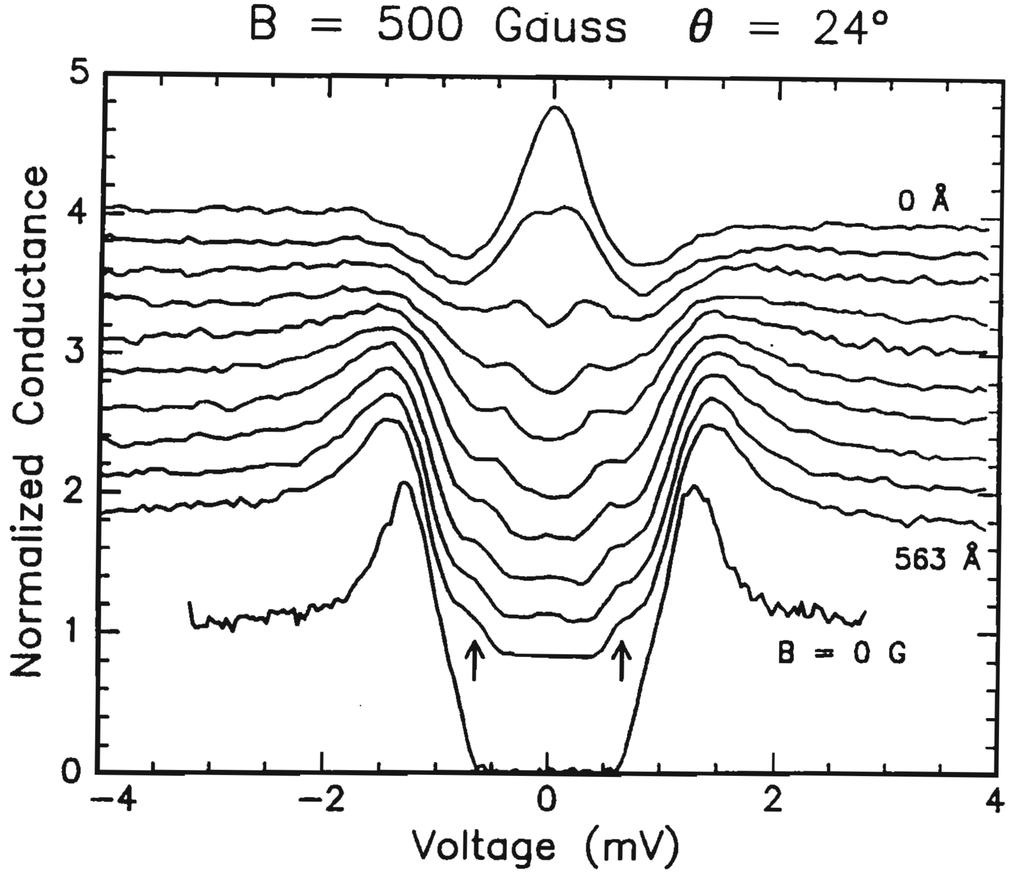

Figure 5.6 Data of Hess *et al.* from Ref. 12. Explicit dI/dV curves from the STM measurement on a $NbSe_2$ sample. The normalized value of 1 corresponds to a metallic tunneling conductance of 5×10^{-9}. The 563 Å curve has been shifted up by 0.75 and successive ones are each shifted by 0.25 normalized units of conductance. The bottom trace shows the spectra at zero magnetic field. The data clearly show that double peak structures were observed at locations away from the center of the vortex.

using the unitary transformation (5.9). It can be shown and understood easily that the quasiparticle wavefunctions $u_n(\mathbf{r})$ and $v_n(\mathbf{r})$ can still be labelled as angular momentum eigenstates and they are all zero at the center of the vortex except for the one with the lowest energy. Therefore, with a finite-range interaction between electrons, the local density of states at the center of the vortex will still be sharply peaked just as in the case with local eletron interactions.

In conclusion, we have discussed the discrepancy between previous theoretical calculations of the vortex core local density of states and a recent STM measurement by Hess *et al.* on a superconducting $\mathrm{NbSe_2}$ sample. We have shown that the Bogoliubov equation formalism can be used to study this problem. We derived the local density of states in terms of the solutions to the Bogoliubov equations. We also solved the Bogoliubov equations numerically for the quasiparticle bound state energies and wavefunctions in the core of a vortex, and have found that the local density of states from our calculation can satisfactorily explain the zero bias tunneling peak seen by Hess *et al.*. We find additional structure in the local density of states, and on the basis of this, we predict that if the tunneling experiment is performed at lower temperatures, the zero-bias peak will split into a double peak (with a dip at zero bias) as the STM tip is moved away from the vortex center. This prediction was indeed verified by new experiment performed at lower temperatures[12]. Through the above discussions and calculations, we conclude that the recent STM experimental observation by Hess *et al.* of the local density of states of a vortex core is in agreement with the BCS superconductivity theory.

Appendix 5A

Tunneling Currents and Density of States

In a scanning-tunneling-microscope (STM) measurement, the electron tunneling occurs at the STM tip. So we expect the tunneling current to be related to the local density of states at the neighborhood of the tip position. We first define $N^>(\omega, \mathbf{r})$ as the density of states at energy ω reached by adding an electron to the system at $\mathbf{r}$, and $N^<(\omega, \mathbf{r})$ as the density of states at energy $(-\omega)$ reached by removing an electron from the system at $\mathbf{r}$. Then on intuitive grounds, when the left side of the tunnel junction is biased at a potential V greater than the right side ($e \equiv 1$), we can write the tunneling currents from left to right as (see reference 1 for detailed discussion of assumptions involved here),

$$I_{L\to R} = A\int_{-\infty}^{\infty} d\omega |T|^2 \left[N_L^<(\omega - V) N_R^>(\omega) - N_L^>(\omega - V) N_R^<(\omega)\right] , \qquad (5A1)$$

where A is a constant of proportionality, V is the applied voltage, and T is the tunneling matrix element which we will assume to be constant for small energy variations. By definition, the density of states $N^>(\omega, \mathbf{r})$ and $N^<(\omega, \mathbf{r})$ can be written as

$$\begin{cases} N_\alpha^>(\omega, \mathbf{r}) = \sum\limits_{n,m} \rho_m \left|\langle n | \psi_\alpha^\dagger(\mathbf{r}) | m \rangle\right|^2 \delta(E_n - E_m - \omega) \, ; \\ N_\alpha^<(\omega, \mathbf{r}) = \sum\limits_{n,m} \rho_m \left|\langle n | \psi_\alpha(\mathbf{r}) | m \rangle\right|^2 \delta(E_n - E_m + \omega) \, , \end{cases} \qquad (5A2)$$

where α represents electron spin and ρ_m is the statistical weight of state m defined in equation (5.2). The density of states $N_\alpha^>(\omega, \mathbf{r})$ and $N_\alpha^<(\omega, \mathbf{r})$ are very simply related. By exchanging the summation index n and m in the expression of $N_\alpha^<(\omega, \mathbf{r})$, we have

$$N_\alpha^<(\omega, \mathbf{r}) = \sum_{n,m} \rho_n \left|\langle m | \psi_\alpha(\mathbf{r}) | n \rangle\right|^2 \delta(E_m - E_n - \omega)$$

$$
\begin{aligned}
&= \sum_{n,m} e^{-\beta\omega} \rho_m \left| \langle n | \psi_\alpha^\dagger(\mathbf{r}) | m \rangle \right|^2 \delta(E_n - E_m - \omega) \\
&= e^{-\beta\omega} N_\alpha^>(\omega, \mathbf{r}) \ . \qquad (5A3)
\end{aligned}
$$

We now define the usual density of states as

$$
\begin{aligned}
N_\alpha(\omega, \mathbf{r}) &\equiv N_\alpha^>(\omega, \mathbf{r}) + N_\alpha^<(\omega, \mathbf{r}) \\
&= (1 + e^{-\beta\omega}) N_\alpha^>(\omega, \mathbf{r}) \\
&= (1 + e^{-\beta\omega}) \sum_{n,m} \rho_m \left| \langle n | \psi_\alpha^\dagger(\mathbf{r}) | m \rangle \right|^2 \delta(E_n - E_m - \omega) \ , \qquad (5A4)
\end{aligned}
$$

and we then have

$$
\begin{cases}
N_\alpha^>(\omega, \mathbf{r}) = N_\alpha(\omega, \mathbf{r})[1 - f(\omega)] \ ; \\
N_\alpha^<(\omega, \mathbf{r}) = N_\alpha(\omega, \mathbf{r}) f(\omega) \ .
\end{cases} \qquad (5A5)
$$

For a normal-superconductor STM tunnel junction with a voltage V applied, the STM tunneling current is

$$
I \approx A|T|^2 \int_{-\infty}^{\infty} d\omega N_n(\omega - V) N_s(\omega) [f(\omega - V) - f(\omega)] \ , \qquad (5A6)
$$

where N_n and N_s denote normal and superconductor density of states, respectively. Assuming that the normal STM tip density of states remains roughly constant for energy variation of size Δ_0, the differential conductance is then

$$
\frac{dI}{dV} \approx A|T|^2 N_n \int_{-\infty}^{\infty} d\omega N_s(\omega + V) \left[-\frac{\partial f(\omega)}{\partial \omega} \right] \ . \qquad (5A7)
$$

The derivative $-\frac{\partial f(\omega)}{\partial \omega}$ is a sharply peaked function with width $k_B T$ and total integral 1. For $k_B T \ll V$, we can treat $-\frac{\partial f}{\partial \omega}$ as a δ-function, and we have

$$
\frac{dI}{dV} \approx A|T|^2 N_n \cdot N_s(V) \ . \qquad (5A8)
$$

This equation is good only with an energy resolution $k_B T$. For temperatures comparable to energy V, one should always use the thermally smeared density of states in equation (5A7) to calculate the differential conductance.

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Chapter 6

Dynamics of the normal to vortex-glass transition: mean field theory

6.1 Introduction

In 1957, Abrikosov[1] showed that a type II superconductor allows partial penetration of magnetic flux in the form of vortices when the external magnetic field is between H_{c1} and H_{c2}. In this *mixed state*, the vortices form a triangular lattice. Later studies showed that the mixed state is not a superconducting state and dissipation of energy occurs when there is an electric current. Two models[2] were introduced to study the transport properties of the mixed state. The flux flow model assumes that the dissipation in a homogeneous type II superconductor is caused by the electric currents flowing through the normal vortex cores drifting under the influence of the Lorentz force. The flux creep model by Anderson and Kim[3] assumes that in an inhomogeneous type II superconductor bundles of vortices can jump between pinning points due to thermal fluctuation and the net flow of vortices transverse to the applied current causes the dissipation of energy. These early transport theories of the mixed state agreed reasonably well with experimental studies on conventional superconductors. But the recently discovered high temperature superconductors have several unusual mixed state properties. These include (1) the existence of an "irreversibility" line in the H-T phase diagram,[4,5] which has the form $T_{\mathrm{irr}} \sim T_0 - H^{2/3}$, reminiscent of spin glass behavior; and (2) an anomalous magnetic field induced broadening of the resistive transition,[6,7,8] which appears to scale as $H^{2/3}$. These new experimental observations demand an explanation based on a thorough study of the dynamics of the mixed state.

Understanding the dynamic properties of the mixed state in type-II superconductors is theoretically challenging due to the competition between collective intervortex interactions, which favor a triangular lattice structure, and vortex pinnings by dirts, which favor a pinned disordered vortex structure. As first discussed by Larkin and Ovchinnikov,[9] pinning destroys the translational long range order of the Abrikosov flux lattice. Recently, Matthew P. A. Fisher[10] argued that pinning and collective effects conspire to produce a vortex glass phase at sufficiently low temperatures. In this phase the vortices are frozen into an equilibrium configuration characterized by a type of "spin glass" order, rather than the translational long range order of the flux lattice.[11] As a result, the linear resistance $R_L \equiv \lim_{I \to 0} V/I$ is identically *zero* in this phase, in contradistinction to the Anderson-Kim model of flux creep,[3] which only considers the effects of pinning and predicts that $R_L \neq 0$ throughout the entire mixed state. Following this, Fisher, Fisher, and Huse[12] developed a scaling theory for the normal to vortex glass phase transition. Tentative experimental evidence for the vortex glass phase in the high temperature superconductor YBCO has been given by Koch *et al.*,[13] and Gammel *et al.*,[14] who have used the proposed scaling theory of the conductivity[12] to interpret the nonlinear I-V characteristics in terms of the vortex glass model. Theoretical evidence for the transition consists of the existence of a vortex glass phase in a two dimensional toy model of a vortex glass,[10,12] and numerical simulations on a simplified three dimensional model of a vortex glass.[15] What remains to be done is a full calculation of the phase boundary, scaling functions, and the critical exponents, which we attempt to study in the following. The work was done in collaboration with Alan T. Dorsey and Matthew P. A. Fisher.[16]

In this chapter we consider a mean field theory for the dynamics of the normal to superconducting vortex glass transition. This allows us to calculate the mean field phase boundary for a realistic model of a vortex glass in three dimensions. In addition, we determine the mean field static and dynamic critical exponents for the transition by calculating the vortex glass order parameter susceptibility. We will also compare the magnetic field induced transition broadening calculated in our mean field theory with that observed in the experiments. Our work draws heavily on studies of the critical dynamics of spin glasses.[17,18]

6.2 The Model

Following Fisher[10], we use for our model the Ginzburg-Landau free energy functional for a superconductor in an external magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$

$$\mathcal{H} = \int d\mathbf{r} \left\{ \frac{1}{2m} |(\nabla - ie^*\mathbf{A})\psi|^2 + [a + a_1(\mathbf{r})] |\psi|^2 + \frac{b}{2} |\psi|^4 - h^*\psi - h\psi^* \right\} \qquad (6.1)$$

where m is the mass of a Cooper pair (assumed to be isotropic), e^* is the electric charge of a Cooper pair, $\hbar = c = 1$, h and h^* are conjugate fields which are introduced to generate response and correlation functions, and where the terms for the magnetic field energy have been dropped. For simplicity we ignore fluctuations in the electromagnetic field. The quenched disorder has been incorporated by defining a random T_c ($a = a_0(T - T_0)$), which is assumed to have Gaussian white noise correlations, so that

$$\begin{cases} \overline{a_1(\mathbf{r})} = 0 \; ; \\ \overline{a_1(\mathbf{r}) a_1(\mathbf{r}')} = 4\Delta\delta(\mathbf{r} - \mathbf{r}') \; , \end{cases} \qquad (6.2)$$

where the overbar denotes an ensemble average over disorder (as throughout the rest of the chapter). This spatial variation in T_c simulates vortex pinning since regions of the sample which have a locally lower T_c, and consequently a higher free energy, will tend to attract the normal cores of the vortices. As argued by

M. P. A. Fisher,[10] at low enough temperatures, the superconductor in an external magnetic field should be in a vortex glass phase with a nonzero Edwards-Anderson order parameter[11] as

$$q_{VG} = \overline{|\langle\psi(\mathbf{r})\rangle|^2} \ , \tag{6.3}$$

where the angular brackets denote a thermal noise average (as throughout the remainder of the chapter).

We are interested in the dynamic properties of the vortex glass phase in order to understand the transport properties of the mixed state. For the dynamics of the disordered Ginzburg-Landau model, we assume relaxational dynamics for the nonconserved order parameter (model A),[19]

$$\Gamma_0^{-1}\partial_t\psi(\mathbf{r},t) = -\frac{\delta\mathcal{H}}{\delta\psi^*(\mathbf{r},t)} + \varsigma(\mathbf{r},t) \ , \tag{6.4}$$

where the thermal noise $\varsigma(\mathbf{r},t)$ is assumed to be a Gaussian white noise:

$$\langle\varsigma^*(\mathbf{r},t)\varsigma(\mathbf{r}',t')\rangle = \frac{2k_BT}{\Gamma_0}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \ . \tag{6.5}$$

Equations (6.1) and (6.4) combine to define our model.

6.3 The mean field theory

As Sompolinsky and Zippelius[17] did with the spin glass dynamics, we use the Martin-Siggia-Rose formalism[20] of writing dynamics models into functional integrals to formulate our mean field theory. The resulting generating function is

$$Z[l,l^*,h,h^*] = \int \mathcal{D}\hat{\psi}\mathcal{D}\hat{\psi}^*\mathcal{D}\psi\mathcal{D}\psi^* J[\psi,\psi^*]e^{\mathcal{L}} \ , \tag{6.6}$$

where

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_R + \mathcal{L}_I \ , \tag{6.7}$$

with

$$\begin{aligned} \mathcal{L}_0 = \int dt d\mathbf{r} \Big\{ & l\psi^* + l^*\psi + h\hat{\psi}^* - h^*\hat{\psi} - \Gamma_0^{-1}|\hat{\psi}|^2 \\ & - \hat{\psi}^*[\Gamma_0^{-1}\partial_t - \frac{1}{2m}(\nabla - ie\mathbf{A})^2 + a]\psi \\ & + \hat{\psi}[\Gamma_0^{-1}\partial_t - \frac{1}{2m}(\nabla + ie\mathbf{A})^2 + a]\psi^* \Big\} \ , \end{aligned} \tag{6.8}$$

$$\mathcal{L}_R = \int d\mathbf{r} a_1(\mathbf{r}) \Big\{ \int dt[\hat{\psi}\psi^* - \hat{\psi}^*\psi] \Big\} \ , \tag{6.9}$$

$$\mathcal{L}_I = \int dt d\mathbf{r} \ b \Big\{ \hat{\psi}\psi^*|\psi|^2 - \hat{\psi}^*\psi|\psi|^2 \Big\} \ , \tag{6.10}$$

$$J[\psi,\psi^*] = \exp\Big\{ \Gamma_0 \int dt d\mathbf{r} \frac{\delta^2 \mathcal{H}}{\delta\psi\delta\psi^*} \Big\} \ , \tag{6.11}$$

where $\hat{\psi}$ and $\hat{\psi}^*$ were introduced to obtain a quadratic $\mathcal{L}_0$. The normalization condition is

$$Z[0,0,0,0] = 1 \ , \tag{6.12}$$

and the thermal average of a general function of the ψ field and its derivatives can be calculated from the generating function as

$$\begin{aligned} \Big\langle f(\psi,\psi^*,\frac{\partial}{\partial h^*},\frac{\partial}{\partial h}) \Big\rangle &= \Big\langle f(\psi,\psi^*,-\hat{\psi},\hat{\psi}^*) \Big\rangle \\ &= f(\frac{\partial}{\partial l^*},\frac{\partial}{\partial l},\frac{\partial}{\partial h^*},\frac{\partial}{\partial h}) Z[l,l^*,h,h^*] \Big|_{l=l^*=h=h^*=0} \ . \end{aligned} \tag{6.13}$$

It is useful to introduce the response function and the correlation function for the ψ field:

$$R(\mathbf{r},t;\mathbf{r}',t') = \frac{\delta\langle\psi(\mathbf{r},t)\rangle}{\delta h(\mathbf{r}',t')} = \Big\langle\, \psi(\mathbf{r},t)\hat{\psi}^*(\mathbf{r}',t')\,\Big\rangle\ ; \tag{6.14}$$

$$C(\mathbf{r},t;\mathbf{r}',t') = \langle\, \psi(\mathbf{r},t)\psi^*(\mathbf{r}',t')\,\rangle\ . \tag{6.15}$$

These are the response and correlation functions for a particular realization of the disorder; the disorder averaged response and correlation functions will be denoted by $\bar{R}$ and $\bar{C}$.

It is very difficult to calculate the generating function Z with the Lagrangian terms (6.7) – (6.10). Here we will make the following mean field approximations. We average over the disorder term $\mathcal{L}_R$ and obtain the following quartic terms

$$\begin{aligned}\bar{\mathcal{L}}_R = \frac{\Delta}{8}\int d\mathbf{r}\int dt_1dt_2\Big[&\hat{\psi}(1)\psi^*(2)\hat{\psi}(2)\psi^*(1) + \hat{\psi}^*(1)\psi(2)\hat{\psi}^*(2)\psi(1)\\ &- 2\hat{\psi}^*(1)\hat{\psi}(2)\psi^*(2)\psi(1)\Big]\ ,\end{aligned} \tag{6.16}$$

where 1 and 2 denote $(\mathbf{r}_1,t_1)$ and $(\mathbf{r}_2,t_2)$ respectively. We then apply the mean field Hartree approximation to the new $\bar{\mathcal{L}}_R$ term and the quartic interaction term $\mathcal{L}_I$. We have

$$\bar{\mathcal{L}}_R \approx \frac{\Delta}{4}\int d\mathbf{r}dt_1dt_2\Big[\bar{R}(2,1)\hat{\psi}^*(2)\psi(1) - \bar{R}^*(2,1)\hat{\psi}(2)\psi^*(1) - \bar{C}(1,2)\hat{\psi}^*(1)\hat{\psi}(2)\Big]\ , \tag{6.17}$$

and

$$\mathcal{L}_I \approx \int d\mathbf{r}dt\ b\Big[\bar{C}(\mathbf{r}t,\mathbf{r}t)\hat{\psi}(\mathbf{r}t)\psi^*(\mathbf{r}t) - \bar{C}(\mathbf{r}t,\mathbf{r}t)\hat{\psi}^*(\mathbf{r}t)\psi(\mathbf{r}t)\Big]\ . \tag{6.18}$$

Putting these two equations back into the generating function, and reverse the procedure of the Martin-Siggia-Rose formalism, we obtain the new effective mean field equation of motion for the field $\psi(\mathbf{r}t)$

$$\begin{aligned}\Gamma_0^{-1}\partial_t\psi(\mathbf{r}t) - \frac{1}{2m}(\nabla - ie\mathbf{A})^2\psi(\mathbf{r}t) + [a + b\bar{C}(\mathbf{r}t,\mathbf{r}t)]\psi(\mathbf{r}t)&\\ - \frac{\Delta}{4}\int dt'\bar{R}(\mathbf{r}t,\mathbf{r}t')\psi(\mathbf{r}t') = \eta(\mathbf{r}t) + h(\mathbf{r}t)\ ,&\end{aligned} \tag{6.19}$$

where the new effective noise $\eta(\mathbf{r}t)$ is no longer a white noise. Instead, the disorder in the system makes it a noise with time memory:

$$\langle\, \eta(\mathbf{r}t)\eta^*(\mathbf{r}'t')\,\rangle = \frac{2k_BT}{\Gamma_0}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') + \frac{\Delta}{4}\bar{C}(\mathbf{r}t,\mathbf{r}t')\delta(\mathbf{r}-\mathbf{r}')\ . \tag{6.20}$$

Notice that the effective equation of motion is linear, we can therefore write

$$\psi(\mathbf{r}t) = \int d\mathbf{r}'dt'\bar{R}(\mathbf{r}t,\mathbf{r}'t')[\eta(\mathbf{r}'t') + h(\mathbf{r}'t')]\ , \tag{6.21}$$

and we can establish a relation between the disorder averaged correlation function $\bar{C}(\mathbf{r}t,\mathbf{r}'t')$ and the response function $\bar{R}(\mathbf{r}t,\mathbf{r}'t')$ as follows,

$$\begin{aligned}\bar{C}(\mathbf{r}t,\mathbf{r}'t') =& \frac{2k_BT}{\Gamma_0}\int d\mathbf{r}_1dt_1\bar{R}(\mathbf{r}t,\mathbf{r}_1t_1)\bar{R}^*(\mathbf{r}'t',\mathbf{r}_1t_1)\\ &+\frac{\Delta}{4}\int d\mathbf{r}_1dt_1dt_2\bar{R}(\mathbf{r}t,\mathbf{r}_1t_1)\bar{R}^*(\mathbf{r}'t',\mathbf{r}_1t_2)\bar{C}(\mathbf{r}_1t_1,\mathbf{r}_1t_2)\\ &+\int d\mathbf{r}_1dt_1d\mathbf{r}_2dt_2\bar{R}(\mathbf{r}t,\mathbf{r}_1t_1)\bar{R}^*(\mathbf{r}'t',\mathbf{r}_2t_2)\,\langle\, h(\mathbf{r}_1t_1)h^*(\mathbf{r}_2,t_2)\,\rangle\ .\end{aligned} \tag{6.22}$$

We can now calculate the vortex glass order parameter susceptibility χ_{VG}. Following the dynamics approach of the spin glass theory by Sompolinsky and Zippelius, we can write the Edwards-Anderson order parameter of the vortex glass phase as the time persistent part of the correlation function $\bar{C}$:

$$\begin{aligned}q_{VG} &= \overline{\langle\,\psi(\mathbf{r}t)\,\rangle\,\langle\,\psi^*(\mathbf{r}t')\,\rangle}\\ &= \lim_{|t-t'|\to\infty}\overline{\langle\,\psi(\mathbf{r}t)\psi^*(\mathbf{r}t')\,\rangle}\\ &= \lim_{|t-t'|\to\infty}\bar{C}(\mathbf{r}t,\mathbf{r}t')\ .\end{aligned} \tag{6.23}$$

As in the case of a spin glass[11], we can also define the vortex glass susceptibility as the response of the vortex glass order parameter with respect to a small random staggered time persistent source field $h(\mathbf{r}t)$ which satisfies

$$\Big[h(\mathbf{r}_1,t_1)h^*(\mathbf{r}_2t_2)\Big]_s = h_r^2\delta(\mathbf{r}_1-\mathbf{r}_2)\ , \tag{6.24}$$

where the square brackets with subscript s denote the average over the random source field. We can then calculate the vortex glass susceptibility using equation (6.22), and we get

$$\chi_{VG} = \frac{q_{VG}}{h_r^2} = \frac{I}{1 - \frac{\Delta}{4} I} , \tag{6.25}$$

where

$$I \equiv \int d(\mathbf{r} - \mathbf{r}_1) |\bar{R}(\mathbf{r}, \mathbf{r}_1; \omega = 0)|^2 . \tag{6.26}$$

We can generalize this vortex glass susceptibility derivation for $\omega \neq 0$ and $\mathbf{k} \neq 0$ by assuming a small random source term with a space and time dependent average as

$$\Big[h(\mathbf{r}_1, t_1) h^*(\mathbf{r}_2 t_2) \Big]_s = h_r^2 e^{i\mathbf{k}\cdot\mathbf{r}_1} e^{-i\omega(t_1 - t_2)} \delta(\mathbf{r}_1 - \mathbf{r}_2) . \tag{6.27}$$

When the time period of the variation of the source term is much larger than the relaxation time τ of the vortex glass system, we can define the vortex glass order parameter as

$$q_{VG} = \lim_{\tau \ll |t - t'| \ll 1/\omega} \bar{C}(\mathbf{r}t, \mathbf{r}t') , \tag{6.28}$$

and we obtain

$$\chi_{VG}(\mathbf{k}, \omega) = \frac{I(k, \omega)}{1 - \frac{\Delta}{4} I(k, \omega)} , \tag{6.29}$$

where

$$I(k, \omega) = \int d(\mathbf{r} - \mathbf{r}_1) e^{-i\mathbf{k}\cdot(\mathbf{r} - \mathbf{r}_1)} |\bar{R}(\mathbf{r}, \mathbf{r}_1; \omega)|^2 . \tag{6.30}$$

The normal to the vortex glass phase transition boundary is then determined by

$$1 - \frac{\Delta}{4} I(k = 0, \omega = 0; T_g) = 0 , \tag{6.31}$$

where T_g is the vortex glass transition temperature. In the following, we will first calculate the shape of the phase boundary $T_g(H)$. We will then calculate all the critical exponents of the normal to vortex glass transition within the mean field theory and express the vortex glass susceptibility in a scaling form.

6.4 The Phase Boundary

The normal to vortex glass transition boundary $T_g(H)$ can be calculated from equation (6.31). To do so, we need to go back to the effective equation of motion (6.19) to calculate the response function. Using the definition of the response function (6.14) and doing Fourier transform of equation (6.19), we obtain the self-consistent equations for the response function $\bar{R}$:

$$\left[-i\Gamma_o^{-1}\omega - \frac{1}{2m}(\nabla - ie\mathbf{A})^2 + \tilde{a}\right]\bar{R}(\mathbf{r},\mathbf{r}';\omega) = \delta(\mathbf{r}-\mathbf{r}') \; ; \tag{6.32}$$

$$\tilde{a} = a + bk_BT\bar{R}(\mathbf{r},\mathbf{r};\omega=0) + bk_BTq_{VG}(\mathbf{r}) - \frac{\Delta}{4}\bar{R}(\mathbf{r},\mathbf{r};\omega) \; , \tag{6.33}$$

where $\tilde{a}$ is the renormalized value of a. In arriving at the above equations, we have rewritten $\bar{C}(\mathbf{r}t,\mathbf{r}t)$ as follows. We first define the physical correlation function with $q_{VG} \neq 0$ as

$$\tilde{C}(\mathbf{r}t,\mathbf{r}'t') = \overline{\langle [\psi(\mathbf{r}t) - \langle \psi(\mathbf{r}t) \rangle][\psi^*(\mathbf{r}'t') - \langle \psi^*(\mathbf{r}'t') \rangle] \rangle} \; , \tag{6.34}$$

from which we have

$$\bar{C}(\mathbf{r}t,\mathbf{r}t) = \tilde{C}(\mathbf{r}t,\mathbf{r}t) + q_{VG}(\mathbf{r}) \; . \tag{6.35}$$

Using the fluctuation-dissipation theorem, we have

$$\bar{C}(\mathbf{r}t,\mathbf{r}t) = k_BT\bar{R}(\mathbf{r}t,\mathbf{r}t) + k_BTq_{VG}(\mathbf{r}) \; . \tag{6.36}$$

We now derive an identity that is useful in determining $I(k=0,\omega=0)$. Taking the derivative with respect to $\tilde{a}$ in equation (6.32), one can easily show

$$I(k=0,\omega=0) = \int d\mathbf{r}_1 |\bar{R}(\mathbf{r},\mathbf{r}_1;\omega=0)|^2 = -\frac{\partial\bar{R}(\mathbf{r},\mathbf{r};\omega=0)}{\partial\tilde{a}} \; . \tag{6.37}$$

We can explicitly express $\bar{R}(\mathbf{r},\mathbf{r};\omega=0)$ in terms of $\tilde{a}$ by solving equation (6.32). We obtain

$$\bar{R}(\mathbf{r},\mathbf{r};\omega=0) = \frac{m^{3/2}\omega_0}{2\sqrt{2}\pi}\sum_{n=0}^{N}\left[(n+\frac{1}{2})\omega_0 + \tilde{a}\right]^{-1/2} \; , \tag{6.38}$$

where $\omega_0 = eH/m$ and N is a cutoff. The phase boundary can then be determined by solving the following two equations simultaneously to determine $T_g(H)$:

$$\sum_{n=0}^{N}\Big[n+\frac{1}{2}+\frac{\tilde{a}_g}{\omega_0}\Big]^{-3/2} = 16\sqrt{2}\pi\delta \ ; \tag{6.39}$$

$$\tilde{a} = a_g + (bk_BT - \frac{\Delta}{4})\frac{m^{3/2}\omega_0^{1/2}}{2\sqrt{2}\pi}\sum_{n=0}^{N}\Big[(n+\frac{1}{2})+\frac{\tilde{a}_g}{\omega_0}\Big]^{-1/2} , \tag{6.40}$$

where $a_g = \alpha(T_g - T_0)$, $\tilde{a}_g$ is $\tilde{a}$ evaluated at the vortex glass transition temperature $T_g(H)$, and δ is a dimensionless parameter defined as

$$\delta \equiv \sqrt{\frac{\omega_0}{\Delta^2 m^3}} \ . \tag{6.41}$$

These two equations are rather difficult in general, however in the limit of high magnetic fields, i.e., $\delta \gg 1$, we need to keep only the lowest Landau level ($n = 0$). In this limit, we obtain,

$$a_g + \frac{1}{2}\omega_0 \approx -\Big(\frac{bk_BT}{\Delta} - \frac{3}{8}\Big)\Big(\frac{m^{3/2}\Delta\omega_0}{\pi}\Big)^{2/3} \ . \tag{6.42}$$

The mean field $T_{c2}(H)$ for the transition from the normal state to the Abrikosov flux lattice state is given by $a_{c2} + \omega_0/2 = 0$, so that the difference is

$$\begin{aligned} a_g - a_{c2} &= \alpha[T_g(H) - T_{c2}(H)] \\ &= -\Big(\frac{bk_BT}{\Delta} - \frac{3}{8}\Big)\Big(\frac{m^{3/2}\Delta\omega_0}{\pi}\Big)^{2/3} \\ &\propto -H^{2/3} \ . \end{aligned} \tag{6.43}$$

Thus, in the limit of high magnetic fields, the difference between the mean field vortex glass phase boundary and the mean field Abrikosov phase boundary scales as $H^{2/3}$. This field dependence is reminiscent of the "irreversibility line" in YBCO which has been discussed by several authors.[4,5] Although our calculated phase boundary $T_g(H) - T_{c2}(H) \sim -H^{2/3}$ is different from the experimental phase boundary $T_{\text{irr}} - T_0 \sim -H^{2/3}$, our calculation may still be able to explain the shape of the experimental irreversibility line if the Abrikosov phase transition temperature $T_{c2}(H)$ does not change too much within the experimental range of variation of the field H.

6.5 The Mean Field Critical Exponents

We now turn to the critical dynamics in the mean field theory. Since we are primarily interested in the long wavelength, low frequency response near the transition, we can expand the vortex glass susceptibility (6.29) for small k, ω, and $\theta \equiv (T - T_g)/T_g$. The expansion can be done by solving equations (6.32) and (6.33) self-consistently. We began by writing the response function as

$$\bar{R}(\mathbf{r},\mathbf{r}';\omega,\theta) = \bar{R}(\mathbf{r},\mathbf{r}';0,0) + \epsilon(\mathbf{r},\mathbf{r}';\omega,\theta) \ . \tag{6.44}$$

Putting this back into equations (6.32) and (6.33), expanding $\epsilon(\mathbf{r},\mathbf{r}';\omega,\theta)$ in the lowest non-trivial order of ω, k, and θ, we obtain

$$\epsilon(\mathbf{r},\mathbf{r}';\omega,\theta) \approx -(\frac{4}{\Delta})^{3/2}\Big[\tilde{u}\theta^2 - i\frac{\omega}{\Gamma_0 A(H)}\Big]^{1/2} \cdot \frac{\int d\mathbf{r}_1 \bar{R}(\mathbf{r},\mathbf{r}_1;0,0)\bar{R}(\mathbf{r}_1,\mathbf{r}';0,0)}{\int d\mathbf{r}_1 |\bar{R}(\mathbf{r},\mathbf{r}_1;0,0)|^2} \ , \tag{6.45}$$

where $\tilde{u}$ is a constant defined as

$$\tilde{u} = \frac{\alpha^2 T_g^2 \Delta^3}{64 b^2} \ , \tag{6.46}$$

and $A(H)$ is defined as

$$\begin{aligned} A(H) &= \int d\mathbf{r}_1 d\mathbf{r}_2 \bar{R}(\mathbf{r},\mathbf{r}_1;0,0)\bar{R}(\mathbf{r}_1,\mathbf{r}_2;0,0)\bar{R}(\mathbf{r}_2,\mathbf{r};0,0) \\ &= \frac{3}{16\sqrt{2}\pi} m^{3/2}\omega_0^{-3/2} \sum_{n=0}^{N} \Big[(n + \frac{1}{2}) + \frac{\tilde{a}_g}{\omega_0}\Big]^{-5/2} \ . \end{aligned} \tag{6.47}$$

Inserting the response function expansion into the vortex glass susceptibility equation (6.29), we finally obtain the scaling function of the vortex glass susceptibility:

$$\begin{aligned}\chi_{VG}(k,\omega) = &\left(\frac{4\sqrt{2}}{\Delta^{3/2}\tilde{u}^{1/2}}\right)\frac{1}{A(H)\theta} \\ &\times \frac{1}{\left[1+\sqrt{1+\frac{\omega^2}{\Gamma_0^2\tilde{u}^2A(H)^2\theta^4}}\right]^{1/2}+\sqrt{\frac{\Delta}{32\tilde{u}}\frac{a_{\parallel}(k_x^2+k_y^2)+a_{\perp}k_z^2}{A(H)\theta}}}\ ,\end{aligned} \tag{6.48}$$

where we have assumed that the external magnetic field is in the z-direction, $a_{\parallel}$ and $a_{\perp}$ are lengths defined as

$$a_{\parallel}^2 = \int d\mathbf{r}_1(x-x_1)^2|\bar{R}(\mathbf{r},\mathbf{r}_1;0,0|^2 = \int d\mathbf{r}_1(y-y_1)^2|\bar{R}(\mathbf{r},\mathbf{r}_1;0,0|^2\ , \tag{6.49}$$

and

$$a_{\perp}^2 = \int d\mathbf{r}_1(z-z_1)^2|\bar{R}(\mathbf{r},\mathbf{r}_1;0,0|^2\ . \tag{6.50}$$

We see that the vortex glass susceptibility satisfies the following scaling form:

$$\chi_{VG} \propto \theta^{-\gamma}X(k\xi,\omega\xi^z)\ , \tag{6.51}$$

with

$$\xi \propto \theta^{-\nu}\ . \tag{6.52}$$

Comparing this scaling form with equation (6.48), we obtain the following mean field critical exponents:

$$\begin{cases}\nu = \dfrac{1}{2}\ ; \\ \gamma = 1\ ; \\ z = 4\ .\end{cases} \tag{6.53}$$

For the critical exponent β, we need to study the temperature region T_g^-. Let the random source term $h_r^2 = 0$ in equation (6.22), taking the limit of $|t-t'| \to \infty$, we obtain

$$q_{VG}[1-\frac{\Delta}{4}I(k,\omega)] = 0\ . \tag{6.54}$$

Since $q_{VG} \neq 0$ below the transition temperature T_g, we again have

$$1 - \frac{\Delta}{4} I(k, \omega) = 0 \; . \tag{6.55}$$

But here we have to solve equation (6.32) self-consistently while keeping the q_{VG} order parameter term. We obtain

$$q_{VG} = -k_B \alpha b \cdot \theta \; , \tag{6.28}$$

from which we have

$$\beta = 1 \; . \tag{6.56}$$

Not surprisingly, all the static and dynamic mean field critical exponents turn out to be the same as those of the spin glass transition. The critical behavior of the normal to vortex glass transition is indeed very similar to that of the paramagnetic phase to spin glass transition.

6.6 Field Induced Transition Width Broadening

As discussed earlier, experimental observations have found[6,7,8] a magnetic field induced broadening of the resistive transition that appears to scale as $H^{2/3}$. If we assume that the low temperature phase in the experiment is indeed the vortex glass phase, we can reasonably explain the field induced broadening within our mean field theory.

The vortex glass susceptibility with $\omega = k = 0$ is

$$\chi_{VG} \propto \frac{1}{A(H)\theta} \, . \tag{6.57}$$

The vortex glass transition width is clearly proportional to $A(H)^{-1}$. We now estimate the field dependence of $A(H)$ in order to estimate the field induced broadening in our mean field theory. Let us go back to the expression of $A(H)$ in equation (6.47). Notice that the rescaled $\tilde{a}_g$ also depends on $\omega_0 = eH/m$, so we need to couple this equation with the phase boundary conditions (6.39) and (6.40). In the high magnetic field limit $\delta = \sqrt{\frac{\omega_0}{\Delta^2 m^3}} \gg 1$, we can keep only the lowest Landau level (n=0) in the summations. Using the result in equation (6.42), we have

$$\begin{aligned} A(H) &\propto \omega_0(\tilde{a}_g + \frac{1}{2}\omega_0)^{-5/2} \\ &\propto \omega_0(\frac{m^{3/2}\Delta\omega_0}{\pi})^{2/3(-5/2)} \\ &\propto \omega_0^{-2/3} \\ &\propto H^{-2/3} \, . \end{aligned} \tag{6.58}$$

So the vortex glass transition width scales as

$$A(H)^{-1} \propto H^{2/3} \, , \tag{6.59}$$

which agrees with the experimental observations.[6,7,8]

6.7 Conductivity

With the mean field value of the dynamic critical exponent $z = 4$, we can now determine the critical behavior of the conductivity in a normal to vortex glass transition. Fisher, Fisher, and Huse[12] developed a scaling theory for the vortex glass transition and their scaling form of the electric conductivity above the transition temperature T_g is

$$\sigma(\omega) \sim \xi^{z-1} S(\omega \xi^{z}) \ . \tag{6.60}$$

The conductivity should then diverge as

$$\sigma \sim \xi^{z-1} \sim \theta^{-\nu(z-1)} \sim \theta^{-3/2} \tag{6.61}$$

as the superconductor approaches the vortex glass transition temperature T_g from the normal phase. This result does not agree well with the experimental measurement which roughly gives[13]

$$\sigma_{\text{exp}} \sim \theta^{-4.8} \ . \tag{6.62}$$

We believe that the discrepancy is due to the fact that the YBCO sample is in the critical regime so the mean field exponents are not expected to be correct. In order to compare well with the experiments, we need to use the renormalization group method to calculate all the critical exponents. We leave this topic to future research.

6.8 Conclusion

In summary, we have formulated a mean field dynamic theory for the normal to vortex glass transition. We obtained all the mean field critical exponents of the transition and determined the phase boundary. We also successfully explained the experimentally observed magnetic field induced transition broadening.

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