Supplementary Materials: Normal form for renormalization groups: The framework for the logs

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MAGNETIZATION

To get the magnetization of 4-d Ising model in the infinite system, we have to solve the following two equations

$$\frac{dt}{d\ell} = 2t - Atu, \tag{1}$$
$$\frac{du}{d\ell} = -u^2 + Du^3. \tag{2}$$

Dividing the two equations by each other gives

$$\frac{dt}{du} = \frac{2t - Atu}{-u^2 + Du^3}, \tag{3}$$

which has the solution

$$\log\frac{t}{t_0} = 2\left(\frac{1}{u} - \frac{1}{u_0}\right) + (2D - A)\log\left(\frac{1/(Du) - 1}{1/(Du_0) - 1)}\right). \tag{4}$$

We want to coarse grain till $t(\ell) = 1$ or equivalently $t(u) = 1$. It also helps to define $s = 1/(Du) - 1$. This is just a convenient variable for calculations. Then

$$-\log t_0 = 2D(s - s_0) + (2D - A)\log s/s_0, \tag{5}$$

where $s_0 = 1/(Du_0) - 1$. This is almost the standard form for the equation of a Lambert-W function defined by $W(z)e^{W(z)} = z$ or equivalently, $\log W(z) + W(z) = \log z$. The solution is

$$s = (2D - A)/(2D)W(xt_0^{1/(A-2D)}), \tag{6}$$

where $x = (2D)/(2D - A)s_0 e^{2D/(2D-A)s_0}$. We also have

$$\frac{du}{-u^2 + Du^3} = dl, \tag{7}$$

which gives

$$\ell = \frac{1}{u} - \frac{1}{u_0} + D\log\left((1/(Du) - 1)/(1/(Du_0) - 1)\right), \tag{8}$$

or

$$\ell = D(s - s_0 + \log s/s_0). \tag{9}$$

We are only interested in the dependence of the magnetization on t_0 because u_0 is an irrelevant variable, and so we can ignore the dependence on it. However, we have to be careful because u is a dangerous irrelevant variable and contributes to the leading singularity of the magnetization. One quick way to get the magnetization is to use the result from mean field theory

$$M \sim \frac{e^{-\ell}}{\sqrt{u}}, \tag{10}$$
$$\sim \frac{e^{-D(s-s_0)}}{D(s+1)}\left(\frac{s}{s_0}\right)^{-D}, \tag{11}$$
$$\sim e^{-Ds}s^{-D}\sqrt{s+1}, \tag{12}$$
$$\sim \exp((A - 2D)/2W(xt_0^{1/(A-2D)}))\times \sqrt{1 + W(xt_0^{1/(A-2D)}})W(xt_0^{1/(A-2D)})^{-D}, \tag{13}$$
$$\sim t_0^{1/2}W(xt_0^{1/(A-2D)})^{-A/2}(1 + W(Yt_0^{1/(A-2D)}))^{1/2}, \tag{14}$$

where we have used the identity $e^{aW(x)} = x^a/W(x)^a$ which follows from the definition of the W function. Finally, near the critical point as $t_0 \to 0$, the W function goes to infinity. So, ignoring the 1, we get

$$M \sim t_0^{1/2}W(xt_0^{1/(A-2D)})^{1/2-A/2}. \tag{15}$$

For the 4-d Ising model, $A = 1/3$, $D = 17/27$, giving

$$M \sim t_0^{1/2}W(Yt_0^{-27/25})^{1/3}, \tag{16}$$

which is the result quoted in the main text.

CHANGING THE LENGTH PARAMETER ℓ IN NORMAL FORM THEORY

The form of the equation for the irrelevant variables u in the 4-d Ising model motivates another change of variables

$$(1 + Du)\frac{du}{d\ell} = -u^2, \tag{17}$$
$$\frac{du}{d\tilde{\ell}} = -u^2 \tag{18}$$

where $\tilde{\ell}$ is defined so $\frac{d\tilde{\ell}}{d\ell} = 1/(1 + Du)$. So far, we have been considering changes of variables in our coordinates but have left the flow parameter ℓ unchanged. This parameter usually corresponds to a physical length or momentum scale. However, there is nothing in principle which stops us from allowing ℓ to depend on the coordinates. This would be somewhat strange from a physical point of view but is not disallowed. The t equation is changed to

$$\frac{dt}{d\tilde{\ell}} = 2t - (A - 2D)tu - 2ADtu^2. \tag{19}$$

However, we can now make another change of variables in t which removes the $2ADtu^2$ term (since that will leave the flow equations for u unchanged) and so the new normal form (after renaming $\tilde{t}$ to t is)

$$\frac{dt}{d\tilde{\ell}} = 2t - (A - 2D)tu. \tag{20}$$

This is a simpler set of equations. Currently, we have not been able to distinguish between these two possible normal forms. This is because for the 4-D Ising model, our scaling form that we derived in the previous section implicitly rely on the equation for L

$$\frac{dL}{d\ell} = -L. \tag{21}$$

We have treated L as a special variable and not included it in our normal form calculations. However, if we include changes in the coarse graining length ℓ, it would be naturally modified to

$$\frac{dL}{d\tilde{\ell}} = -L(1 + Du). \tag{22}$$

Since the extra term is a resonance, it cannot be removed by a change of variables. In effect, this leaves du/dL and dt/dL unchanged. Hence, this analysis does not seem to matter for the finite scaling analysis of the 4-d Ising model. However, it is quite possible that such an analysis could be carried out in other cases where it does fundamentally change the form of the scaling and then it would be interesting to test it.