UNIVERSAL PROPERTIES OF THE TRANSITION FROM QUASI-PERIODICITY TO CHAOS IN DISSIPATIVE SYSTEMS

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An exact renormalization group transformation is developed for dissipative systems which describes how the transition to chaos may occur in a continuous and universal manner if the frequency ratio in the quasi-periodic regime is held at a fixed irrational value. Our approach is a natural extension of K.A.M. theory to strong coupling. Most of our analysis is for analytic circle maps. We have found a strong coupling fixed point where invertibility is lost, which describes the universal features of the transition to chaos. We find numerically that any two such critical maps with the same winding number are C¹ conjugate. It follows that the low frequency peaks in an experimental spectrum are universal and we determine how their envelope scales with frequency.

When the winding number has a periodic continued fraction, our renormalization transform has a fixed point and spectra are self similar in addition. For a set of non-periodic winding numbers with full measure our renormalization transformation yields an ergodic trajectory in a sub-space of all critical maps. Physically one finds singular and universal spectra that do not scale.

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1. Introduction

In this paper we show how a quasi-periodic flow, with two sharp incommensurate frequencies, can be made to bifurcate to a chaotic or turbulent state in a quantitatively universal way. The transition we envisage is a continuous one and our analyses make essential use of renormalization group methods which were first introduced in the context of dynamical systems by Feigenbaum [1]. There are now many experiments in low aspect ratio systems that observe quasi-periodic behavior prior to the onset of turbulence [2]. Usually, however, the actual transition is preceded by mode locking in which the ratio of the two basic frequencies, \( \omega_1 \) and \( \omega_2 \), sticks at a rational value for some range of parameters. The observed flow is then periodic.

A number of reasonable conjectures can then be made about how the invariant 2-torus in phase space on which the quasi-periodic motion takes place breaks down and gives rise to turbulence. Although the actual events that precede this transition are quite complex (and are left for an appendix), in the absence of period doubling there do not appear to be any quantitative and universal features that could be measured in a real experiment. Thus it is difficult to test these conjectures.

We therefore discuss how to modify the experiments, in a tractable way, so as to make the transition from quasi-periodicity to turbulence occur directly and yield a clear experimental signature. Our proposal requires that two experimental parameters be controlled in a consistent way. (In the jargon, the transition is then said to be of co-dimension two.) Typically one just varies one (i.e., the Rayleigh or Reynolds' number) which explains why this transition has not been seen before.

Since it will emerge that the transition to turbulence approached in this way is universal, it is sufficient to study only the simplest nontrivial dynamical model. Clearly the model must possess an attracting invariant 2-torus to describe quasi-periodicity. Theoretically, it is then convenient to look at successive intersections of this flow with a suitably chosen surface (the Poincaré section) and to work only with maps. Since the universal features of the transition reside only in the long time behavior of the system no essential information is thereby lost. In an experiment in which the second frequency, \( \omega_2 \), is introduced by means of a periodic external force, the same reduction could be accomplished by observing the quasi-periodic system every \( 2\pi/\omega_2 \) sec. On the map, the invariant torus becomes an invariant circle.

A model system rich enough to include the transition we intend to study is the following family of invertible analytic maps of the annulus:

\[
P_{\omega,a}(r, \phi) = (1 + \lambda(r - 1) - (a/2\pi) \sin(2\pi \phi), \\
\phi + \omega + \lambda(r - 1) - (a/2\pi) \sin(2\pi \phi)),
\]

(1.1)

where \( 0 < \lambda < 1 \) and \( r \) and \( \phi \) are polar coordinates. It will be useful to note that for infinite contraction (\( \lambda = 0 \)), (1.1) reduces to an analytic map of the circle

\[
\phi' = \phi + \omega - (a/2\pi) \sin(2\pi \phi).
\]

(1.2)

The remaining two parameters are both relevant to our analysis; \( a \) controls the nonlinearity and is akin to the Reynolds' number, and \( \omega \) sets the rotation rate. By the latter we mean just the mean rotation rate of \( \phi \) per iteration which in the quasi-periodic regime is just \( \omega_1/\omega_2 \). (A precise definition is given below.)

When \( a = 0, \ r = 1 \) is an attracting invariant circle and \( P_{\omega,a} \) is a pure rotation with rotation number \( \omega \). In fact, this circle is normally hyperbolic [3] and it may therefore be proven that for small values of \( a \) this circle persists although the flow on it may be mode-locked. Since \( P_{\omega,a} \) contracts areas uniformly, (its Jacobian is \( \lambda \)), any invariant curve is unique. Let \( \rho = \rho(\omega, a) \) be the rotation number of the restriction \( f_{\omega,a} \) of \( P_{\omega,a} \) to this invariant circle. We now ask how this curve breaks down as the nonlinearity is increased.

To refine this question we need to consider the
bifurcation structure in the \((\omega, a)\) parameter space. For small coupling, \(0 < a \ll 1\), the picture follows from the small divisor perturbation theory of Kolmogorov, Arnold and Moser [4, 5, 6]. The regions \(I_{p/q}\) of parameter space where \(\rho(\omega, a) = p/q\) (\(p, q\) integers) and where \(f_{\omega,a}\) has an orbit of period \(q\) are tongues of the general form shown in fig. 1. Each tongue intersects the line \(a = \text{const.} > 0\) in a nontrivial closed interval. Between the tongues are curves of the form \(\omega = u(a)\), \((u\) a continuous function) on which \(\rho\) is irrational and where \(f_{\omega,a}\) is conjugate through a continuous change of variables to a trivial rotation. The union of these curves has positive Lebesque measure. The \(I_{p/q}\) are the phase locked regions.

One lesson of small divisor theory is that it is sensible to use the parameter \(\omega\) to control \(\rho\) so that we can increase the coupling \(a\) and keep \(\rho\) fixed at a suitable irrational value.

Under these circumstances there is good evidence (to be presented here and in ref. 11) for the existence of a critical value of \(a = a^*\) such that for \(a < a^*\) there is an analytic invariant circle, which becomes nonanalytic at \(a = a^*\) and ceases to exist for \(a > a^*\). The radius of convergence shrinks smoothly to zero as \(a\) approaches \(a^*\) from below and our renormalization group is designed to extract the universal scaling properties of the critical curve at \(a = a^*\). Knowledge of the conjugacy which relates the singular invariant curve to the pure rotation is equivalent to determining the long time behavior of iterates of the map and thus the low frequency peaks in an experimental spectrum. One result of our analysis will be to show that all the frequency peaks are universal and to calculate their relative amplitudes at \(a = a^*\).

The analysis necessary to extract the universal data at \(a = a^*\) encoded in high iterates of the annular map is enormously simplified by realizing that we can restrict our attention to maps of the circle. The reduction in dimension from an annulus to a circle is ultimately due to the dissipation. Of course the annular map could be expanding in some region provided only that the average contraction rate along the invariant curve is positive. The contraction rate of the \(n\)th iterated map then approaches \(n\)-times the mean contraction rate. More precisely, there will be uniform exponential convergence onto the invariant curve along directions that intersect it transversely. Thus the annular map is only nontrivial in the direction along the invariant curve. For \(a \ll a^*\) one can sensibly define an analytic and invertible circle map as the restriction of the annular map (or some fixed iterate thereof) to the invariant curve.

Unfortunately for our purposes this simple construction is not uniformly appropriate as \(a \rightarrow a^*\).

To understand the difficulty, assume the mechanism for the smooth loss of analyticity at \(a^*\) is a tangency between the contracting direction and the invariant curve. Baring accidental symmetries there should be a preferred phase \(\phi^*\) where the angle between the two is minimal as \(a \rightarrow a^*\). Examination of a high iterate of the map in the limit of zero angle reveals zero slope of \(\phi^*\) in the associated circle map. The difficulty in our construction is that to achieve contraction onto the invariant curve requires an infinite number of iterates which spread the points of zero derivative, generically inflections, densely around the circle. We would like to claim that the circle map so obtained is equivalent to performing the same iterations on an
analytic circle map with a single point of zero slope at $\phi^*$. This delicate point as far as we can see can only be established with the aid of the renormalization group developed here. The complication of an infinite number of inflection points is avoided by dilating, expanding and truncating the domain of definition after each iteration so as to retain only the inflection point at $\phi^*$. The difficult step is to show that iterates of the annular map under the same process of dilating and truncating are described by the same limiting function as characterizes circle maps. The limiting annular map has the desired property that the foliation is tangent to the invariant curve at only one point in the reduced domain and further that the contraction rate in one iteration is infinite. To make the presentation coherent we develop our formalism for circle maps first and defer applications to annular maps to the very last section.

In the next section we discuss some of what is known mathematically about analytic homeomorphisms of the circle (e.g., (1.2)). We then ask how large iterates of the map scale. When the inverse homeomorphism is differentiable the answer to this question follows simply from deep results of Arnold and Herman [4, 7, 8]. However, when the map in question develops a cubic inflection point (i.e., just at the point where it becomes noninvertible) a different scaling is found. This is reflected in the structure of the nonanalytic invariant circle at the onset of chaos. Section 2 concludes with a synopsis of our numerical results for inflectional maps which were found independently by Shenker [9, 10, 11].

In section 3 we define a renormalization group which reduces questions of scaling and universality to the existence of a fixed point with certain stability properties. In section 4 we solve for this fixed point numerically and discuss the flows induced by our renormalization group transformation. By linearizing around the fixed point we recover the exponents previously found by directly iterating the map.

In section 5, as a prerequisite to examining spectra, we establish a connection between the functional transformation generated by our renormalization group and the more conventional description in terms of a conjugate homeomorphism familiar from K.A.M. theory. A number of precise conjectures concerning the degree of smoothness of homeomorphisms relating maps with an inflection point are formulated. The universal features expected in the Fourier spectrum of an experimental time-series are identified in section 6. The overall scaling properties of such spectra are also demonstrated.

In section 7 we consider situations in which our renormalization transformation has no fixed point yet there are singular and universal spectra. The notion of an ergodic renormalization group trajectory in function space and its Liapunov exponents is discussed. Section 8 indicates how to extend our renormalization group to two-dimensional maps such as (1.1). In the conclusion, we consider how to experimentally test our predictions and mention several other small divisor problems that may be treated by renormalization group methods. Related results have been independently obtained by Feigenbaum, Kadanoff and Shenker [11].

The appendix collects mathematical results for non-invertible maps of the circle and annular maps with particular attention to the qualitative question of how the transition from a mode locked state to chaos occurs when the rotation number is not controlled.

2. Maps of the circle

Firstly, we discuss the well known mathematical theory for diffeomorphisms of the circle. Then we describe the scaling relations obtained in numerical experiments for analytic maps of the circle with a single inflection point which is cubic (we call these cubic critical maps). We leave to section 4 the explanation of these relations in terms of a renormalisation transformation and to section 6 and the
Appendix: The justification of the connection with the transition from quasi-periodicity.

2.1. Generalities and the rotation number

We shall represent the circle $T^1$ as the real numbers mod 1 and denote the real line as $\mathbb{R}$. A homeomorphism of $T^1$ (resp. $\mathbb{R}$) is a continuous mapping of $T^1$ (resp. $\mathbb{R}$) onto itself with a continuous inverse. An analytic (resp. $C^r$, $1 \leq r \leq \infty$) diffeomorphism is an analytic (resp. $C^r$) homeomorphism with an analytic (resp. $C^r$) inverse. Every homeomorphism of $T^1$ can be represented by a homeomorphism $f$ of $\mathbb{R}$ such that $f(\theta + 1) = f(\theta) + 1$; the associated circle homeomorphism is then $\theta \rightarrow f(\theta) \mod 1$. We denote the class of such homeomorphisms of $\mathbb{R}$ by $\mathcal{D}_0$.

The rotation number of $f \in \mathcal{D}_0$ is

$$\rho(f) = \lim_{n \to \infty} n^{-1}(f^n(\theta) - \theta). \quad (2.1)$$

This limit exists and is independent of $\theta$. Another useful characterisation is this:

$$\sigma = \rho(f) \iff |f^n(\theta) - \theta - n\sigma| < 1, \text{ for all } n, \theta. \quad (2.2)$$

(In terms of the original quasi-periodic flow, $\rho$ is the ratio of two fundamental frequencies—consequently it is accessible experimentally.)

Some elementary facts about $\rho$ are these:

a) if $R_\delta(\theta) = \theta + \delta$, then $\rho(R_\delta) = \delta$;

b) if $f \in \mathcal{D}_0$ and $\rho(f) = p/q$ then there exists $\theta$ such that $f^q(\theta) = p + \theta$ and $\theta \mod 1$ is a periodic point of the associated circle homeomorphism;

c) $\rho$ depends continuously on $f \in \mathcal{D}_0$ in the $C^0$-topology: this means that, given $f$, if $\epsilon > 0$ there exists $\delta > 0$ such that $|f(\theta) - g(\theta)| < \delta$ implies $|\rho(f) - \rho(g)| < \epsilon$.

d) if $f_\mu$ is the 1-parameter family given by $f_\mu = R_\mu \cdot f$ then $\rho(f_\mu)$ is an increasing function of $\mu$ and is strictly increasing at points where $\rho$ is irrational [7].

2.2. Conjugation to a rotation

Consider the map $\theta \rightarrow f(\theta)$ where $f \in \mathcal{D}_0$. On making the coordinate change $\theta = h(\phi)$, $h \in \mathcal{D}_0$, this map becomes $\phi \rightarrow h^{-1}fh(\phi)$. We say that $f$ and $g$ are conjugate (resp. analytically conjugate) if there exists a homeomorphism (resp. analytic diffeomorphism) $h$ such that $g = h^{-1}fh$. It immediately follows from (2.2) that two conjugate homeomorphisms have the same rotation number. A basic problem is to determine when the converse holds and, in particular, to determine when $f \in \mathcal{D}_0$ is conjugate (or better analytically conjugate) to $R_\sigma(\theta)$. Then every orbit of $R_\sigma$ is dense in $T^1$. And $f$ is conjugate to $R_\sigma$ if $f$ has an orbit $\{f^n(\theta)\}_{n=0}^\infty$ which is dense in $T^1$ because then we can define $h$ by the condition that $h(n\sigma) = f^n(\sigma)$. Denjoy [12] has shown that if $\log f'$ is of bounded variation and $\sigma = \rho(f)$ is irrational then every orbit of $f$ in $T^1$ is dense. This condition is satisfied by $C^2$ diffeomorphisms with irrational rotation number.

Note that the above argument shows that if $h$ exists then it is unique up to a translation. Thus for a given $f$ one can ask how smooth $h$ is. The following basic result about this is due to Herman [7]: there exists a set $\mathcal{A} = [0, 1]$ with Lebesgue measure one such that, if $f$ is an analytic diffeomorphism and $\sigma = \rho(f) \in \mathcal{A}$, then $f$ is analytically conjugate to $R_\sigma$. For our purposes it is sufficient to note that $\mathcal{A}$ contains those numbers with bounded entries in their continued fractions.

2.3. Continued fractions and scaling

We now consider analytic diffeomorphisms $f$ with irrational rotation number $\sigma = \rho(f)$. Since we are interested in the rational approximations of $\sigma$ it is convenient to express it as a continued fraction:

$$\sigma = \frac{1}{n_1 + \frac{1}{n_2 + \ldots}} = [n_1, n_2, n_3, \ldots]. \quad (2.3)$$

Within the set of all rational numbers whose denominator does not exceed a given bound, the
best rational approximation to \( \sigma \) is

\[
p_{m/q_m} = [n_1, n_2, \ldots, n_m, n_{m+1} = \infty].
\]

We use this notation henceforth. For the rest of this section we restrict ourselves to the case where \( n_{m+s} = n_m \) for some \( s \geq 1 \). This condition is imposed so that we can speak of self similarity for appropriate iterates of the map.

Define \( \tau = \lim_{s \to \infty} q_{n+s}/q_n \). When \( s = 1 \), \( \tau = \sigma^{-1} \).

By Herman's Theorem \([7]\) we have that \( f \) is analytically conjugate to \( R_\sigma \), where \( \sigma = \rho(f) \), i.e.,

\[
f^{-s} = h R_{\sigma,s} h^{-1} \quad \text{with } h \text{ an analytic diffeomorphism.}
\]

Consequently, \( f^{q_n} - p_n = h(R_{\sigma,s} - p_n)h^{-1} \) converges to the identity as \( n \to \infty \). Moreover, for large \( n \),

\[
f^{q_n}(0) - p_n \text{ is closer to 0 than } f^{q_n}(0) - l \text{ for all } 1 \leq k < q_n \text{ and } l = 0, 1, 2, \ldots. \]

Because the points \( f^{q_n}(0) - p_n \) and \( R_{\sigma,s}(0) - p_n \) are related by this analytic coordinate change;

\[
\alpha = \lim_{n \to \infty} (f^{q_n}(0) - p_n)/(f^{q_{n+s}}(0) - p_{n+s}) = \lim_{n \to \infty} (R_{\sigma,s}(0) - p_n)/(R_{\sigma,s+}\sigma(0) - p_{n+s}) = (-) \tau.
\]

Let \( 1/\alpha(n) = (f^{q_n}(0) - p_n)/\sigma \) and let \( \lambda_n(\theta) = a\theta \).

Note that \( \lim_{n \to \infty} (\alpha(n)/\alpha(n)) \) exists and is nonzero.

**Lemma 2.1.** As \( n \to \infty \), \( \lambda_n(f^{q_n} - p_n)\lambda_n^{-1} \) converges as an analytic function to \( R_\sigma \) on any bounded domain.

**Proof.** Let \( B_\delta \) denote \( \{z \in \mathbb{C} : |\text{Im } z| < \delta \} \). Since \( f \) and \( h \) are analytic there exists \( \delta > 0 \) and analytic extensions \( \tilde{f} \) and \( \tilde{h} \) of \( f \) and \( h \) to \( B_\delta \) so that for \( z \in B_\delta \),

\[
\tilde{f} \circ \tilde{h}(z) = \tilde{h}(z + \sigma).
\]

We have to prove that

\[
\lim_{n \to \infty} \sup_{z \in B_\delta} |\alpha(n)(f^{q_n}(z/\alpha(n)) - p_n) - R_\sigma(z)| = 0,
\]

where \( A \) serves to delimit an arbitrary finite interval independent of \( n \).

Let \( z/\alpha(n) = \tilde{h}(y) \) and observe that

\[
f^{q_n}(\tilde{h}(y)) - p_n = \tilde{h}(y) + \tilde{h}'(y)(q_n\sigma - p_n) + O(\alpha^{-2n}).
\]

Elimination of \( y \) in favor of \( z \) together with the observation that \( \text{sup}(y) \to 0 \) as \( n \to \infty \) implies

\[
\alpha(n)(f^{q_n}(z/\alpha(n)) - p_n) - R_\sigma(z) = \alpha(n)(q_n\sigma - p_n)h'(0) - \sigma + O(\alpha^{-n}).
\]

The definition of \( \alpha(n) \) then insures that the right-hand side is \( O(\alpha^{-n}) \).

Q.E.D.

It is worth noting that to obtain a universal limit for \( f^{q_n} \) and in particular \( R_\sigma \), the scale factor itself had to depend on \( f \), i.e., we used \( \alpha(n) \) and not \( \alpha \). When we consider cubic critical maps we again will have to make a non-universal scale change to obtain a universal limiting function which replaces \( R_\sigma \) in (2.5). The analogue of \( \lim_{n \to \infty} (\alpha(n)/\alpha(n)) = \alpha \) will again exist and be universal.

A second scale \( \delta \) can be defined by asking how quickly diffeomorphisms with a \( p_n/q_n \)-cycle approach \( f \). More precisely, let \( f_\lambda = R_\sigma f \) and let \( \lambda_n(\theta) \) be the value of \( \lambda \) closest to 0 such that \( \rho(f_\lambda) = p_n/q_n \). Then if \( f \) is an analytic diffeomorphism (with \( \rho(f) = \sigma \) of course), it follows from Herman's results \([7]\) that

\[
\delta = \lim_{n \to \infty} \lambda_n(\theta)/\lambda_n+1(\theta) = -\tau^2.
\]

In the special case where the family is given by (1.2),

\[
\delta = \lim_{n \to \infty} (\omega_{n+1} - \omega_n)/(\omega_n - \omega_n-1), \quad (2.6)
\]

where for a given value of \( a \) (\(|a| < 1\)), it is convenient to define \( \omega_n \) as the value of \( \omega \) such that 0 is contained in a \( q_n \)-cycle with rotation number \( p_n/q_n \). (The same definition of \( \delta \) is used for \(|a| = 1\).) Thus \( \delta \) measures how quickly the phase-locked tongues \( I_{p/q} \) converge onto the curve \( \rho(f_{\omega_n}) = \sigma \) in \( 0 < \omega < 1, |a| < 1 \). A third rescaling factor \( \gamma \) will be
needed in sections 4–5 to describe how maps that are nearly critical renormalize toward the weak coupling fixed point.

The notation \( \delta \) was chosen in partial analogy to Feigenbaum’s study [1] of period doubling where a single exponent described both the accumulation of periodic \( 2^n \) cycles and the onset of chaos. Our problem is codimension two (i.e., there are two relevant directions) which requires us to define both \( \delta \) and \( \gamma \).

2.4. Scaling for cubic critical maps

We now consider the scaling relations for cubic critical maps analogous to (2.4) and (2.5). These were discovered by us, and independently, by Shenker, in computer experiments mainly on the family (1.2) [9, 10]

\[
\theta \rightarrow f_\omega(\theta) = \theta + \omega - (a/2\pi) \sin(2\pi \theta). \tag{2.7}
\]

Our conventions are arranged such that (2.7) has a cubic critical point at 0 for \( a = 1 \). We will look for scaling around the origin since it is known from general theorems (appendix) that knowledge of the orbit of the critical point largely determines the dynamics of the map. The numerical method used is similar to that of Greene [14]. We worked mainly with the irrational winding number that is most poorly approximated by rationals, \( \sigma = \sigma_0 = (\sqrt{5} - 1)/2 \) for which \( n_1 = 1 \) and \( q_1 = 1, q_2 = 2; q_{n+1} = q_n + q_{n-1} \) and \( p_n = q_{n-1} \).

By solving the equation \( f_{\omega_n}(0) = p_n \) for \( \omega_n \) we can estimate \( \delta \) from its definition (2.6). With 14-figure arithmetic and \( \sigma = \sigma_0, \omega_n \) reached its limiting value for \( n = 28 \) \( (q_{28} = 514229) \). The corresponding value of \( \alpha \) was estimated from the analogue to (2.4), i.e.

\[
\alpha = \lim_{n \to \infty} (f_{\omega_n}^{-1}(0) - p_{n-1})/(f_{\omega_n}(0) - p_n). \tag{2.8}
\]

In the same calculation we also determined the function

\[
\zeta(x) = \lim_{n \to \infty} x^{(n)}(f_{\omega_n}(x/x^{(n)}) - p_n), \tag{2.9}
\]

where the scale factor \( x^{(n)} \) was determined by insisting that \( \zeta(0) = 1 \) for each approximate \( n \).

The finite \( n \) approximations to \( \alpha \) and \( \zeta \) continued to improve out to \( n = 17 \) \( (q_{17} = 2584) \). From the fluctuations between successive cycles we estimate the error in \( \zeta \) (fig. 2) to be \( 2 \times 10^{-5} \) using the \( L_1 \) norm over the interval bounded by the two inflection points that bracket the origin. We also found \( \alpha = -1.28862 \pm 0.00005 \) and \( \delta = -2.83361 \pm 0.00001 \). We believe our limiting error reflects just the loss of accuracy due to the cubic inflection point together with a machine accuracy of \( 10^{-14} \). Table I contains a synopsis of our numerical results for other winding numbers and maps. From this one sees that \( \zeta \) as well as the scale factors \( \alpha \) and \( \delta \) depend on the rotation number and the order of the critical inflection point (e.g., cubic) but are otherwise universal and independent of other details of the map.

2.5. Fixed point equations (\( \sigma = \sigma_0 \))

From the definition of \( \zeta \) in (2.9) and the relations

\[
f_{\omega_n+1} = f_{\omega_n} \circ f_{\omega_n-1} = f_{\omega_n-1} \circ f_{\omega_n}
\]

one sees that \( \zeta \) must
The scaling factors \( \alpha \) and \( \delta \) for the critical sine map (2.7) and the power map (2.11) with a cubic inflection point and for the winding numbers indicated. Note that the scale factor corresponding to a full period of the continued fraction is \( \alpha' \) by (2.4). The convergence to a universal value with the order of the rational approximate is clearly oscillatory for both \( \alpha \) and \( \delta \). No attempt to model the error term was made although it might have added one significant digit to our values. The quoted error is large enough to bracket three successive values of the parameter. We also verified, although with lesser accuracy, that for \( \rho = \sigma_0 \) when \( \hbar \to 1 \) in (2.11) \( \alpha \) tends smoothly to \( 1/\alpha \). Specifically for \( \hbar = 2, 1.3, 1.1, \) and \( 1.02 \), \( \alpha = 1.388 \pm 0.002, 1.474 \pm 0.002, 1.5817 \pm 0.0002, 1.6103 \pm 0.0002 \). The corresponding \( \delta \) values are \( 2.7078, 2.6494, 2.61978, 2.618052 \pm 0.000002 \) for \( \hbar = 1.01 \). In the opposite limit of \( \hbar \to \alpha \), \( \alpha \to 1 \), e.g., for \( \hbar = 4, 8, 16 \), \( \alpha = 1.231 \pm 0.002, 1.131 \pm 0.005 \) and \( 1.071 \pm 0.002 \).

\[
\begin{array}{lll}
\text{Equation (2.7)} & \\
\rho & 1, 1, 1, \ldots & 2, 2, 2, \ldots & 1, 2, 1, 2, \ldots \\
\alpha & 1.28862 \pm 0.00005 & 1.5868 \pm 0.0001 & 1.4032 \pm 0.0002 \\
\delta & 2.83361 \pm 0.00001 & 6.79925 \pm 0.00005 & 17.66906 \pm 0.00002 \\
\end{array}
\]

\[
\begin{array}{lll}
\text{Equation (2.11)} & \\
\rho & 1, 1, 1, \ldots & 2, 2, 2, \ldots & 1, 2, 1, 2, \ldots \\
\alpha & 1.2885 \pm 0.0003 & 1.5865 \pm 0.0005 & 1.4033 \pm 0.0002 \\
\delta & 2.833 \pm 0.002 & 6.795 \pm 0.005 & 17.670 \pm 0.005 \\
\end{array}
\]

satisfy the equations

\[
\zeta(\theta) = \alpha \zeta(\alpha \zeta(\theta/\alpha^2)) \tag{2.10a}
\]

and

\[
\zeta(\theta) = \alpha^2 \zeta(\alpha^{-1} \zeta(\theta/\alpha)) \tag{2.10b}
\]

Feigenbaum et al. [11] have numerically solved (2.10) and found nontrivial solutions. However, (2.10) is rather confusing because, as we shall see from our renormalisation group analysis, (2.10b) is essentially redundant and (2.10a) alone specifies \( \zeta \) up to a scale change [15]. Also, this approach becomes very cumbersome when one deals with other winding numbers as the equations corresponding to (2.10) become more numerous and more complicated. The renormalisation transformation constructed in the next section provides a clear computational tool for calculating \( \zeta \) as well as an explanation of its existence (cf. (3.6)).

2.6. Other critical maps

We have also investigated a number of non-analytic functions of the form

\[
\theta \to f(\theta) = \omega + \theta |2\theta|^{b-1}, \tag{2.11}
\]

where \( \theta \in [-\frac{1}{2}, \frac{1}{2}] \). In each case we obtain good convergence for \( \alpha, \delta \) and \( \zeta \) but now these depend on the value of \( b \). The entries in table I for \( b = 3 \) are a check that \( \alpha \) and \( \delta \) are universal for generic critical maps even when the map in question is not an analytic homeomorphism (e.g., (2.11)).

Jonker and Rand [16] have proven by means of the renormalization group transformation in the following section that \( \alpha \) and \( \delta \) are analytic functions of \( b - 1 = \epsilon \) for small \( \epsilon \). This makes unambiguous the numerical determination of the leading coefficient in an \( \epsilon \)-expansion. Specifically for \( \rho = \sigma G, -\alpha = 1/\sigma G - 0.37 \epsilon \) and \( -\delta = \sigma G^2 + 0.19 \epsilon^2 \) with errors of order 5% in the coefficients.
3. Renormalization group analysis of circle homeomorphisms

We now give an explanation of the above phenomena by implementing a renormalisation group analysis along the general lines developed by Feigenbaum in his study of period doubling [1, 17]. Our construction is slightly more complicated than Feigenbaum’s since it depends upon rotation number and our orbits are not periodic. Nevertheless, our transformation is defined on an open set of maps that does not constrain the winding number.

The great utility of the renormalisation group in dynamical systems has been to reduce questions of existence and universality to a geometric problem in function space, which, for period doubling, has been brought under sufficiently quantitative control to permit mathematical proofs [17, 18, 19]. An additional virtue of the renormalisation group for our problem is that the universal features of the spectra will follow from the coordinate change that reduces our fixed point homeomorphism $f_*$ to a rotation. There is no need to work backwards from $f_*$, which controls the iterates near the inflection point, to recover the rest of the orbit.

Although our main interest is with the class of cubic critical mappings of the circle (i.e., analytic maps with a unique inflection point which is cubic), it turns out that to construct a renormalisation transformation we have to work in a larger space of suitably constrained pairs of analytic homeomorphisms of the line. After studying the renormalisation transformation in this bigger space, we obtain the scaling properties etc. of the analytic mappings by considering how they are embedded in this space and how the transformation acts upon them.

Each element of this bigger space naturally defines a piecewise analytic map of the circle. This is extremely useful because it allows an immediate definition of rotation number and more importantly enables us to construct conjugating homeomorphisms which play an important role in our demonstration of the universality of the spectrum. These conjugacies possess unexpected properties which are shared by both analytic and our piecewise analytic circle mappings.

The renormalisation group transformation $T_n$ applied to a particular circle homeomorphism $f$ (and the space we embed it in) depends upon $n = n_1$, where

$$\rho(f) = \frac{1}{\frac{1}{n_1} + \frac{1}{n_2} + \cdots} = [n_1, n_2, \ldots],$$

i.e., $n_1$ is such that $n_1 \leq 1/\rho(f) < n_1 + 1$. In this way arbitrary rotation numbers can be studied because the image of $f$ under $T_n$ will have rotation number $1/(n_2 + 1/(n_3 + \cdots))$. When the continued fraction (3.1) is periodic, say $n_{i+s} = n_i$, we seek a fixed point of

$$T = T_{n_{i+1}} \circ \cdots \circ T_{n_1}$$

and study the structure of $T$ near the fixed point (its linearisation, stable and unstable manifolds, etc.). Associated with this fixed point is the universal 2-parameter family of circle maps which correspond to its unstable manifold.

3.1. Definitions and formal properties [20]

We now define $T_n$ and the space $\mathcal{S}_n$ upon which it acts. Let $\mathcal{S}_n$ consist of the pairs $(\xi, \eta)$ of analytic homeomorphisms of $\mathbb{R}$ which satisfy the following conditions ($'= derivative):$

(a) $\xi(0) = \eta(0) + 1$,
(b) $\eta'(\xi(0)) = \xi'(\eta(0))$,
(c) $0 < \xi(0) < 1$,
(d) $\xi'(\eta(0)) > 0$,
(e) $\xi^{s-1}(\eta(0)) < 0$,  \hspace{1cm} (3.3)
(f) if $\xi'(x) = 0$ or $\eta'(x) = 0$ for $x \in [\eta(0), \xi(0)]$ then $x = 0$ and $\eta'(0) = \xi'(0) = \eta''(0) = \xi''(0) = 0$, but $\xi''(0)$ and $\eta''(0)$ are nonzero;
and
\[(g) \quad (\xi \eta)'(0) = (\eta \xi)'(0), \quad \text{and if } \xi'(0) = 0 \]
then
\[(\xi \eta)''(0) = (\eta \xi)''(0).\]

Let \(S_{\text{crit}}\) denote the subset of those \((\xi, \eta)\) in \(S_n\) with \(\xi'(0) = \eta'(0) = 0\). If \(\sigma = [n, n_2, \ldots]\) let \(S_{\sigma, \text{crit}}\) denote those \((\xi, \eta)\) in \(S_{\sigma, \text{crit}}\) with \(\rho(\xi, \eta) = \sigma\). Note that (a), (b) and (c) imply \(\eta(0) < \xi(\eta(0)) < \xi(0)\) and condition (e) is redundant when \(n = 1\).

Conditions (a)–(c) permit us to associate a homeomorphism \(f = f_{\sigma, \epsilon}\) on the unit circle with each pair \((\xi, \eta)\in S_n\). Define \(f = \xi\) on \([\eta(0), 0]\) and \(f = \eta\) on \([0, \xi(0)]\) and associate the unit interval \([\eta(0), \xi(0)]\) with the circle by identifying end points. Fig. 3 illustrates our construction. A rotation number \(\rho(\xi, \eta) = \rho(f)\) can be defined for \(f\) in the usual way. Conditions (d)–(e) ensure that \(n < 1/\rho(f) < n + 1\). Let \(S_n\) be the space of homeomorphisms obtained from \(S_n\) in this manner.

Now define a mapping \(T_n\) on \(S_n\) by

\[T_n(\xi, \eta) = (\alpha \xi^{-1} \eta \alpha^{-1}, \alpha \xi^{-1} \eta \alpha^{-1}). \quad (3.4)\]

The scale factor \(\alpha = 1/(\xi^{-1} \eta(0) - \xi \eta(0))\) obeys \(\alpha < -1\) by (d)–(e). Note that the compositions involved are all well defined. This is not the case if \(\xi\) and \(\eta\) are interchanged in the right-hand side of (3.4). It is easy to verify that conditions (a)–(c) and (f)–(g) are preserved by \(T_n\). Condition (f) restricts us to cubic critical points and condition (g) eliminates an uninteresting marginal direction at the fixed point (see section 4) and is satisfied by all those \((\xi, \eta)\) which came from analytic circle mappings since they commute.

The analytic diffeomorphisms and cubic critical maps \(f \in S\) are embedded in our space \(S\) as follows: with \(f\) we associate the pair \((\xi, \eta) = (f, f^{-1})\).

Let \(\tilde{T}_n\) be the mapping induced by \(T_n\) on \(S\). The action of \(\tilde{T}_n\) is most conveniently illustrated for the case \(n = 1\) (fig. 4). If \(f = f_{\epsilon, \sigma} \in S = S_1\), then let \(f(0^+) = \eta(0)\) and \(f(0^-) = \xi(0)\) and define

\[g = \begin{cases} f^2, & \text{on } [f(0^+), 0], \\ f, & \text{on } [0, f(0^-)]. \end{cases}\]

Then \(\tilde{T}_1 f = \alpha g x^{-1}\) where \(\alpha = 1/(f(0^+) - f^2(0^-))\).

Returning to the case of arbitrary \(n\) it may be
shown that if \( f = f_{\xi,\alpha} \), then \( \tilde{T}_n^k(f) \) is given by
\[
(\beta^k f_{\alpha} \beta^{-k}, \beta^k (f_{\alpha} + \alpha - 1) \beta^{-k}),
\]
where \( \beta \) is the geometric mean of the first \( k \) scalings \( \alpha \).

3.2. Behaviour of the rotation number under \( T \)

From this construction it may also be shown that \( \rho(T f) = (1/\rho(f)) - 1 \). Divide the domain of \( f \) into three regions \( I_1 = [f(0^+), 0] \), \( I_2 = [0, f(0^-)] \), and \( I_3 = [f(0^-), 0] \) then \( f(I_1) = I_3 \) and \( f(I_3) \subseteq I_1 \cup I_2 \). Consider the orbit under \( f \) of any point \( x \) that avoids 0. The fraction of points on this orbit that fall in \( I_1 \), and thus in \( I_3 \), is precisely \( 1 - \rho(f) \). But from an orbit of \( f \) we can construct one for \( g \) by eliminating all points in \( I_3 \). Thus the map \( T(f) \) has a fraction of \( (1 - \rho(f))/\rho(f) \) positive elements on its orbit.

Q.E.D.

The same argument effectively applies to arbitrary \( T_n \) acting on \( \mathcal{S}_n \).

Lemma 3.1. If \( \rho(\zeta, \eta) = 1/(n + 1/(n_2 + \cdots)) \) then \( \rho(T_n(\zeta, \eta)) = (1/\rho(\zeta, \eta)) - n \).

Proof. Let \( I_0 = [0, \xi(0)], I_l = [\xi^{l-1}(0), \xi^l(0)], l = 1, \ldots, n, I_1 = [\xi^{n-1}(0), 0] \) and \( I_2 = [0, \xi^n(0)] \). Choose \( x \) in \( I_2 \) such that \( f^k(x) \neq 0 \) for all \( k \geq 0 \) \((f = f_{\xi,\alpha})\) and let \( m(k) \) denote the number of elements of \( x, f(x), \ldots, f^{k-1}(x) \) which fall into \( I_0 \). Then \( \rho(f) = \lim_{k \to \infty} m(k)/k \).

Now choose a sequence \( k_i \) such that \( k_i < k_{i+1} \) and \( f^{k_i}(x) \in J_1 \). Let \( \gamma = \alpha x, 1/\alpha = \xi^n - 1 \) \( (0) - \xi^n \eta(0) \). Then there are precisely \( m(k) \) elements of the sequence \( x, f(x), \ldots, f^{k_i}(x) \) in \( I_0 \) and hence the same number in \( I_1 \), \( l = 1, \ldots, n \). Thus there are \( k_i = nm(k_i) \) in \( J_1 \). Consequently, \( k_i = nm(k_i) \) points of the sequence \( y, (f(y))^{-1}(y), \ldots, (f(y))^{m(k_i)-1}(y) \) fall in \( I_0(\tilde{T}f) \) which proves that
\[
\rho(\tilde{T}f) = \lim_{k_i \to \infty} (k_i - nm(k_i))/m(k_i) = \frac{1}{\rho(f)} - n.
\]

Q.E.D.

We note the following three immediate corollaries of the Lemma for the case \( n = 1 \) and \( \sigma = \sigma_G = [1, 1, \ldots] \) although there are obvious analogues for other periodic continued fractions:

(a) \( \rho(f) = q_{-1}/q_1 \) implies \( \rho(\tilde{T}f) = q_{-2}/q_{-1} \);
(b) \( \rho(f) = \sigma_G \) if and only if \( \rho(\tilde{T}f) = \sigma_G \); (3.5)
(c) if \( T_n(\zeta, \eta) \in \mathcal{S}_1 \) for all \( k \geq 0 \), then \( \rho(f_{\xi,\alpha}) = \sigma_G \).

The last statement follows from the transformation properties of \( \rho \) and the observation that \( (\zeta, \eta) \in \mathcal{S}_1 \) implies \( 1 < \rho^{-1}(\zeta, \eta) < 2 \).

From the above lemma, it only makes sense to search for a fixed point of our renormalisation group transformation when the continued fraction of \( \rho(f) \) is eventually periodic. Any integers that precede the periodic part may be removed by application of appropriate \( T_n \). Only the tail of the continued fraction is relevant in determining which nontrivial fixed point might exist. A number of remarks about the application of \( T_n \) to maps with nonperiodic winding numbers are reserved for section 7.

3.3. Fixed points of \( T \)

If \( (n_1, n_2, \ldots, n_r) \) are the integers in one period of the continued fraction of \( \sigma = \rho(f) \), then one should look for a fixed point in \( \mathcal{S}_{n_i} \) of the renormalisation transformation \( T \) defined by (3.2). Actually \( T \) must possess a weak-coupling fixed point corresponding to rotation by \( \sigma \) and we believe a second nontrivial fixed point on \( \mathcal{S}_{\sigma,\text{crit}} \) (which we call the strong-coupling fixed point) corresponding to the onset of chaos at the breakdown of an invariant torus with rotation number \( \sigma \). The strong-coupling fixed point and associated eigenvalues are found numerically in the following section for two cases \( n_i = 1 \) and \( n_i = 2 \).

Our picture is as follows (fig. 5): The strong-coupling fixed point \( (\zeta_*, \eta_*) \) \((\rho(\zeta_*, \eta_*) = \sigma)\), has a two-dimensional unstable manifold (corresponding to two eigenvalues \( \delta \) and \( \gamma \) of the linearization of \( T \) at \( (\zeta_*, \eta_*) \)) which contains the curve
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See the full page for the complete text.
and the associated stable manifold has important implications for the orbits of cubic critical maps with the prescribed rotation number. We state our Theorem only for \( \sigma = \sigma_G \) though the construction obviously generalizes to other winding numbers along the lines of lemma 3.1. The extension to general periodic \( \sigma \) requires a lower bound on the slope of the fixed point over an interval, which we have only in a compelling way for \( \sigma = \sigma_G \).

**Theorem 3.1.** Let \( f \) be in the stable manifold of \( f_\ast = (\xi_\ast, \eta_\ast) \) with \( \rho(f) = \sigma_G = \sigma \) then it is conjugate to a rotation.

**Proof.** If \( f \) is not conjugate to the rotation \( R_\ast \) there is a nontrivial closed interval \( I \) such that \( f^n I \cap I = \emptyset \) for all \( n \geq 0 \) [7]. We call such intervals Denjoy intervals. Assume that \( f \) possesses such an interval and let \( I(f) \) denote the length of a longest such interval. Note that since \( f \) is in the stable manifold, the origin cannot be in a Denjoy interval because as \( n \to \infty, f^n(0) \) approaches 0 from both sides. Consequently, \( f(0) \) and \( f^2(0) \) are not in a Denjoy interval. Let \( I(f) \) be a Denjoy interval of maximal length. Then \( I(f) \) must be contained in one of the three open intervals on the circle with end-points \( 0, f^2(0) \) and \( f(0) \). If \( f \) is sufficiently close to \( f_\ast \) it cannot be in \( (f^2(0), f(0)) \) because \( fI \) is also a Denjoy interval which is longer than \( f \) because the slope of \( f_\ast \) is greater than \( 2\eta_\ast \). But then we have that \( \alpha \) is a Denjoy interval for \( \tilde{f} \). This shows that for all \( f \) in some neighbourhood of \( f_\ast \),

\[
I(\tilde{f}(f)) > 0.9|\alpha_*| I(f).
\]

This leads to a contradiction since on iteration we get that \( I(\tilde{f}^n(f)) \) is greater than one for \( n \) sufficiently large.

The result now follows because, if \( f \) is in the stable manifold and \( f \) has a Denjoy interval, then by similar arguments \( \tilde{f}^n(f) \) has a Denjoy interval and for large \( n \), \( \tilde{f}^n(f) \) will be in the above neighbourhood.

### 4. The fixed point and its eigenvalues

In this section we discuss the eigenvalues and eigenvectors of the linearisation of the renormalisation transformation \( T \) at the two fixed points, and the computation of the strong-coupling fixed point. For clarity we restrict the discussion to the case \( \sigma = \sigma_G = (\sqrt{5} - 1)/2 \) so that \( T = T_1 \). All of the ideas presented here generalise immediately to arbitrary rotation numbers with periodic continued fractions.

#### 4.1. Formal properties of eigenvalues of \( T \)

Let \( \Xi_\ast = (\xi_\ast, \eta_\ast) \) denote the strong-coupling fixed point of \( T \). For an appropriate choice of the space of analytic pairs \((\xi, \eta)\) the derivative of \( T \) at \( \Xi_\ast, dT_\ast = dT(\Xi_\ast) \) is a compact operator [16]. Consequently, its spectrum consists of a countable set of eigenvalues with no accumulation point different from zero. There is no continuous spectrum.

When analyzing the spectrum of \( dT_\ast \) it is convenient to replace \( T \), which involves a scaling factor which depends upon \((\xi, \eta)\) by a transformation involving only a constant scaling factor. The price is the introduction of one eigenvalue equal to 1 and some change in the eigenfunctions. To do this we decompose \( T \) as

\[
T = PC,
\]

where, if \( \beta_{\xi, \eta} = \xi(0) - \eta(0), \)

\[
P(\xi, \eta)(x) = \beta_{\xi, \eta}^{-1}(\xi, \eta)(\beta_{\xi, \eta} x)
\]

and

\[
C(\xi, \eta)(x) = \alpha_*(\eta, \eta \xi)(x/\alpha_*).
\]

where

\[
\alpha_* = \frac{1}{\eta_* \xi_*(0) - \eta_* \xi_*(0)}.
\]
Since $\mathcal{E}_*$ is a fixed point for $T$ and $C$ we have $P(\mathcal{E}_*) = \mathcal{E}_*$. Also, $P^2 = P$ and

$$PC = PCP.$$  \hspace{1cm} (4.3)

**Lemma 4.2.** The eigenvalues $\lambda \neq 1$ of $dT_* = dT(\mathcal{E}_*)$ and $dC_* = dC(\mathcal{E}_*)$ are identical together with their multiplicities. Let $Z_* = (x\xi_*(x) - \xi_*(x), x\eta_*(x) - \eta_*(x))$, then it is an eigenvector of $dC_*$ with eigenvalue 1 but $dT_*Z_* = 0$.

**Remark.** We will demonstrate numerically that 1 is not in the spectrum of $dT_*$ so the restriction to $\lambda \neq 1$ in lemma 4.2 is superfluous.

**Proof.** The spectrum of $dP_* = dP(\mathcal{E}_*)$ consists of 0 and 1. The eigenvalue 0 has multiplicity one since its eigenvector $Z = (X, Y)$ satisfies

$$Z = -(X(0) - Y(0))Z_* .$$

We establish the second assertion of the lemma first. Observe that

$$Z_* = \frac{d}{dt} s_*(\xi_*, \eta_*) \bigg|_{t=0},$$

where $s_*(\xi, \eta)(x) = (1 + t)^{-1}(\xi, \eta)((1 + t)x)$. Then

$$Cs_* = s_* \quad \text{and} \quad Ts_* = Ts_0 .$$

Differentiating then implies $dC_*Z_* = Z_*$ and $dT_*Z_* = 0$.

Now if $Z$ is an eigenvector of $dC_*$ with eigenvalue $\lambda$; then by (4.3) $dP_*Z$ is an eigenvector of $dT_*$ and if $dP_*Z \neq 0$, $\lambda$ is an eigenvalue. Conversely, if $Z \neq Z_*$ is an eigenvector of $dT_*$ then so is $dP_*Z$ and by (4.3)

$$dC_*Z = Z + cZ_* .$$

But setting $Z = \tilde{Z} + cZ_*(1 - \lambda)$ proves $\lambda$ is an eigenvalue of $dC_*$. Q.E.D.

We now prove a number of results about the eigenvalues of $dT_*$ and $dC_*$ which will be useful when we come to interpret the numerical results. Before proceeding we note the following three equalities each of which follows by differentiating the fixed point equations (and the two additional equations obtained by composing the two sides of $\mathcal{E}_* = T(\mathcal{E}_*)$), three times and utilising $\xi'(0) = \xi'_*(0) = 0$:

$$\xi'(0) = \alpha^2,$$  \hspace{1cm} (4.4)
 $$\eta'(0) = \alpha^4,$$  \hspace{1cm} (4.5)
 $$\eta(\xi_*(0)) = \alpha^2 .$$  \hspace{1cm} (4.6)

First we consider which eigenvalues can correspond to eigenvectors which are not tangent to the subspace of commuting functions $\xi \eta = \eta \xi$.

**Lemma 4.3.** If $(X, Y)$ is an eigenvector of $dC_*$ tangent to $(\xi \eta - \eta \xi)(0) = 0$ for $k < v$ and not tangent to $(\xi \eta - \eta \xi)(0) = 0$ then the associated eigenvalue is $-\alpha^{3-v}$.

**Proof.** Let $F(\xi, \eta) = \xi \eta - \eta \xi$. Then if $Z = (\xi, \eta)$ a simple calculation using $s_*(\xi, \eta)_* = s_*\xi_*$ shows that

$$d(F \cdot C)(\mathcal{E}_*) \cdot (X, Y)(x) = -\alpha \eta_*(\eta_*(\xi_*^*(x/\alpha)) \cdot dF(\mathcal{E}_*) \cdot (X, Y)(x/\alpha) .$$

But by (4.6) $\eta_*(\eta_*(\xi_*^*(0)) = \alpha^2$ so the result follows by differentiating both sides $v$ times at 0. Q.E.D.

Note that lemma 4.3 tells us that any eigenvector whose eigenvalue is not a power of $\alpha$ is tangent to the commuting subspace.

We now consider eigenvectors which are not analytic functions of $x^3$.

**Lemma 4.4.** If $(X, Y)$ is an eigenvector of $dC_*$ on $dT_*$, then either $X$ and $Y$ are analytic functions of $x^3$ or the associated eigenvalue is $\lambda = \pm \alpha^{3-p}$ where $p$ is the least integer which is not a multiple of 3 such that the $p$th-derivative $X^{(p)}(0) \neq 0$. In the
latter case the sign of \( \lambda \) is \((-)^{p+1} \text{sign} \) \((X^{(p)}(0) \cdot Y^{(p)}(0))\).

**Proof.** By the lemma 4.2 we need only prove this for \(C\). Let \((X, Y)\) be an eigenvector which is not a function of \(x^3\) and let \(p\) be as above. If \(\lambda\) is the eigenvalue,

\[
\begin{align*}
\lambda X(x) &= \alpha Y(x/\alpha), \quad (4.7a) \\
\lambda Y(x) &= \alpha \eta^*(\xi^*(x/\alpha)) \cdot X(x/\alpha) + \alpha Y(\xi^*(x/\alpha)). \quad (4.7b)
\end{align*}
\]

In (4.7b) the terms \(\alpha Y(\xi^*(x/\alpha))\) and \(\alpha \eta^*(\xi^*(x/\alpha))\) are analytic functions of \(x^3\) since \(\xi^*\) is. Consequently, we deduce that

\[
\lambda Y^{(p)}(0) = \alpha^{1-p} \eta^*(\xi^*(0)) X^{(p)}(0). \quad (4.8)
\]

Thus differentiating (4.7a) \(p\) times at 0 and substituting in (4.8) we obtain

\[
\lambda^2 = \eta^*(\xi^*(0)) \alpha^{2-2p}.
\]

But \(\eta^*(\xi^*(0)) = \alpha^4\) by (4.5). Q.E.D.

Of particular interest is the amount by which a small linear term added to the fixed point is amplified by iteration of the renormalisation transformation. This describes the “cross-over” from critical maps to diffeomorphisms. Later, we will obtain information about the structure of conjugacies from our knowledge of this “cross-over”.

**Lemma 4.5.** If \((X, Y)\) is an eigenvector of \(dC_\star\) or \(dT_\star\) tangent to \((\xi \eta - \xi' \eta')(0) = 0\) and such that \(X(0) = Y(0)\) but \(X'(0) \neq 0\) or \(Y'(0) \neq 0\) then the associated eigenvalue is \(\gamma = \alpha^2\) and \(\text{sign}(X'(0)) = \text{sign}(Y'(0))\).

**Remark.** We will later give evidence for the existence of such an eigenvector.

**Proof.** By lemma 4.4, \(\gamma = \pm \alpha^2\). By combining the linear equation obtained from the condition that \((X, Y)\) is tangent to \((\xi \eta - \eta \xi')(0) = 0\) with

\[
dC_\star \cdot (X, Y) = \lambda (X, Y)
\]

we obtain

\[
Y'(0)[\xi^*(\eta^*(0)) - \lambda] = 0
\]

and \(Y'(0) \neq 0\). Therefore \(\lambda = \xi^*(\eta^*(0)) = \alpha^2\). Q.E.D.

Finally, we note that if we let

\[
s_t(\xi, \eta) = (1 + t\sigma)^{-1} \cdot (\xi, \eta) \cdot (1 + t\sigma),
\]

then

\[
Z(\sigma) = \frac{d}{dt} s_t(\xi^*, \eta^*) = (\xi^* \sigma - \sigma \xi^* \eta^* - \sigma \eta^*)
\]

is an eigenvector of \(dC(\xi^*, \eta^*)\) if

\[
\alpha \sigma (x/\alpha) \equiv \lambda \sigma (x)
\]

and then the associated eigenvalue is \(\lambda\). In particular, the coordinate changes corresponding to \(\sigma(x) = x^a\) give eigenvectors of \(dC\) with eigenvalue \(\lambda = \alpha^{1-n}\). The only eigenvalues \(\lambda\) with \(|\lambda| \geq 1\) produced in this way correspond to \(n = 0\) and 1, i.e., to a shift of origin and a change of scale. Recall that the eigenvector for the latter lies in the kernel of \(dP_\star\) and therefore yields an eigenvalue of 0 for \(dT_\star\) (lemma 4.2).

### 4.2. Calculation of fixed points

We now describe how the strong-coupling fixed point \((\xi^*, \eta^*)\) of our renormalisation transformation \(T\) was obtained numerically. We mainly deal here with the case where \(T = T_1\) so that \(\sigma = \sigma_0\). It will be clear how to generalise the procedure so as to treat \(\sigma = \rho(f)\) with an arbitrary periodic continued fraction.

Because of the nonlinear nature of \(S\) and \(S_{\text{crit}}\) it is numerically convenient to relax the conditions...
3.3(b) and (g), extend $T$ to the bigger space and then remove those eigenfunctions which violate these conditions. In lemma 3.2 we proved that the strong coupling fixed point $(\xi_\ast, \eta_\ast)$ is an analytic function of $x^3$ if it exists. We therefore only consider the action of $T$ on the subspace consisting of those $(\xi, \eta)$ which are analytic functions of $x^3$. This subspace is invariant under $T$. We approximate $(\xi, \eta)$ by

$$
\xi(x) = \sum_{n=0}^{N-1} a_n x^{3n},
$$

$$
\eta(x) = \sum_{n=0}^{N-1} b_n x^{3n}.
$$

For a fixed $N$ we can define on the space $\mathbb{R}^{2N}$ of coefficients $a_i, b_i$ an approximation, $\hat{T}$, to $T$ by means of the following equations:

$$
\sum_{n=0}^{N-1} a_n (x_m^R)^{3n} = \alpha \eta (x_m^L/\alpha), \quad (4.10a)
$$

$$
\sum_{n=0}^{N-1} b_n (x_m^L)^{3n} = \alpha \eta \xi (x_m^L/\alpha), \quad (4.10b)
$$

where

$$
1 \leq m \leq N,
$$

$$
x_m^R = \left(\frac{m - 1}{N - 1}\right)^{1/3} \alpha \eta (\xi(0)),
$$

$$
x_m^L = \left(\frac{m - 1}{N - 1}\right)^{1/3} \alpha \eta (0),
$$

and

$$
\alpha^{-1} = \eta(0) - \eta \xi(0). \quad (4.10c)
$$

The idea of distributing the matching points $x_m^R, x_m^L$ in this way is due to Feigenbaum [11]. Now both $\alpha$ and the righthand side of (4.10b) are nonlinear functions of the coefficients $a_i, b_i$ from the last approximation to $(\xi, \eta)$. Inversion of the coefficient matrix formed by the powers of $x^L, x^R$ on the left-hand side of (4.10) yields the new $(a_i, b_i)$ and completes the definition of $T$.

Newton’s method may then be applied to find a fixed point of this nonlinear system. With 14-figure numerical accuracy the convergence ceased to improve for $N$ beyond 11 and ultimately Newton’s method would not converge. For $N = 11$ however the fixed point equation is satisfied to within $10^{-7}$ over the entire interval. The series for $(\xi, \eta)$ however does not appear to converge over the entire interval since the contribution of the last term at the boundary is only a factor of 10 smaller than the first few terms. We suspect that there are singularities of $(\xi, \eta)$ in the complex plane that determine the radius of convergence of a Taylor series and some other expansion point than 0 should be used [21]. However, since we can accurately approximate a solution of the fixed point equations in the space $(a_i, b_i)$ we can also compute the relevant eigenvalues of $dT_\ast$ with comparable accuracy.

The differentiation $dT(a_\ast, b_\ast)$ was done numerically. (It is as easy to compute $dC(a_\ast, b_\ast)$ by omitting eq. (4.10c) and fixing $\alpha$ at its fixed point value.) For $N = 11$ we find three eigenvalues $\lambda$ with $|\lambda| \geq 1$;

$$
\delta = -2.83362 \pm 2 \times 10^{-5} = -\sigma_G^{2.16444},
$$

$$
\lambda_1 = 2.1395796,
$$

$$
\lambda_2 = -0.99997,
$$

while from the fixed point calculation

$$
\alpha = -1.288575 \pm 15 \times 10^{-6} = -\sigma_G^{0.5204718}.
$$

The error bars on $\alpha$ and $\delta$ are somewhat subjective and based on comparison between the approximations for various $N$.

We believe that $\lambda_1$ and $\lambda_2$ actually equal $-\alpha^3$ ($= 2.139583$) and $-1$. These are expected by lemma 4.3 since we have not imposed the conditions $(\xi \eta - \eta \xi)^v(0) = 0$ for $v = 0, 3$ on the eigenvectors and indeed we find numerically that the eigenvectors associated with $\lambda_{1,2}$ violate their re-
spective conditions by an amount of order 1. The agreement between $\lambda_{1,2}$ and $-\alpha^2$, $-1$ is an additional check on the quality of our numerics. We also infer (and verified) that since $\delta$ is not a power of $\alpha$, lemma 4.3 implies it must be tangent to $\xi_0(x) = \eta_0(x)$.

The leading irrelevant eigenvalue (in $S_{\text{crit}}$) is determined to far less accuracy and we estimate $|\lambda_3| = 0.52 \pm 0.02$.

We therefore conclude this: The spectrum of $\mathbf{d}T(\xi_*, \eta_*)$ restricted to the tangent space of $S_{\text{crit}}$ consists of an eigenvalue $\delta = 2.83362$ and a countable number of eigenvalues $\lambda$ such that $|\lambda| < 0.54$.

Let $R$ denote the set of those $(\xi, \eta)$ in $S_{\text{crit}}$ such that $\rho(\xi, \eta) = \sigma_G$. It is clear that the stable manifold of $(\xi_*, \eta_*)$ in $S_{\text{crit}}$ is contained in $R$. We shall assume that near $(\xi_*, \eta_*)$, $R$ is contained in the stable manifold. (Merely knowing the eigenvalue spectrum of $\mathbf{d}T$ is not sufficient to exclude the possibility that there are small regions of the $R$ manifold close to the fixed point but not in the stable manifold. This is known not to occur for maps with an $\epsilon$-singularity for $\epsilon \geq 0$ small.

Then we deduce that the eigenspace associated with $\delta$ is transverse to $R$. This is because $R$ is of co-dimension one in $S_{\text{crit}}$ and the compact operator $\mathbf{d}T_*$ leaves invariant its tangent space. Consequently, there is a one-dimensional subspace complementary to the tangent space of $R$ which is invariant under $\mathbf{d}T_*$. Clearly, this must be the eigenspace corresponding to $\delta$.

With some mild assumptions on the nature of the sets $\rho = p_n/q_n$ (see ref. 16) it now follows (by an argument similar to that given by Collet, Eckmann and Lanford [17]) that if $(\xi_\mu, \eta_\mu)$ is a one-parameter family in $S_{\text{crit}}$ which is transverse to $R$ and if $\mu_k$ is such that 0 is a periodic point with rotation number $p_k/q_k$ of the circle map defined by $(\xi_\mu_k, \eta_\mu_k)$ then

$$\lim_{j \to \infty} \delta(\mu_j) - \lim_{k \to \infty} \mu_k$$

exists and is non-zero.

We now argue that $\mathbf{d}T_*$ has one other eigenvector $\gamma$ which is transverse to $S_{\text{crit}}$, tangent to $\rho = \sigma_G$ and has an eigenvalue $\alpha^2$. We know from general principles [22] and find from numerical experiments that $(\xi_*, \eta_*)$ is not attracting under $T$ restricted to $\rho = \sigma_G$. For example, we find if $f_\mu(\theta) = \theta + \omega_\mu - (a/2\pi) \sin 2\pi \theta$ with $\omega_\mu$ chosen so that $\rho(f_\mu) = \sigma_G$, then $T^n(f_\mu - (a/2\pi))$ converges as $n \to \infty$ to a function near $(\xi_*, \eta_*)$ (see section 5). Also by the results described above we know that $\mathbf{d}T_*$ has no eigenvalue $\lambda$ with $|\lambda| = 1$. Thus $\mathbf{d}T_*$ must have an eigenvector transverse to $S_{\text{crit}}$ in $S$ with an eigenvalue $\gamma$ such that $|\gamma| > 1$. By lemma 4.4 one must then have that $\gamma = \pm \alpha^{3-p}$ where $p = 1, 2$. The case $p = 2$ is not possible, for then the eigenvector is not tangent to $\xi''(0) = 0 = \eta''(0)$. Thus we deduce that $p = 1$. Then it follows from lemma 4.5 that $\gamma = \alpha^2$.

The same numerical procedures have been implemented for the case where the rotation number $\sigma = (\sqrt{2} - 1) = [2, 2, 2, \ldots]$. Again, in the space of functions analytic in $x^3$, we obtain for $N = 8$ three eigenvalues $\lambda$ with $|\lambda| \geq 1$:

$$\delta = -6.79924 \pm 2 \times 10^{-5} = -\sigma^{-0.17480},$$

$$\lambda_1 = 3.99563,$$

$$\lambda_2 = -0.9999989,$$

and from the fixed point

$$\alpha = 1.586822 \pm 2 \times 10^{-6} = \sigma^{-0.523879}$$

we again infer that $\lambda_1 = \alpha^2 = 3.99562$ and $\lambda_2 = -1$.

By relaxing condition 3.3(f), a number of numerical experiments were done in which $T_1$ was applied to a cubic critical map with the inflection point not at the origin. No fixed point was found and the position of the inflection point moved randomly around the unit interval.

4.3. Eigenvalues at the weak-coupling fixed point

The weak-coupling fixed point for $T = T_1$ is
given by

\[(\xi_*, \eta_*)_n(x) = (x + \sigma_0, x - \sigma_0^2)\],

and the associated value of \(\alpha\) is \(-1/\sigma_0\). For the rest of this section we will work with \(C\) rather than \(T\).

If \((X, Y)\) is an eigenvector of \(dC\) then

\[\lambda Y(x) = \alpha Y(\xi_*(x/\alpha)) + (\alpha^2/\lambda) Y(x/\alpha^2)\].

Therefore, if we take \(Y(x) = \sum_{n=0}^{N} a_n x^n\) we find that

\[\lambda = \alpha^{-N} \quad \text{or} \quad -\alpha^{2-N}\].

There are thus four eigenvalues \(\lambda\) of \(dC\) with \(1 > 1\) namely \(+1\), \(-\alpha\) and \(-\alpha^2\). The associated number \(\sigma_0\) are conjugate to \(R_{\sigma_0}\). We now consider in detail the structure of the homeomorphism \(h\) which conjugates a cubic critical map \(f\) with irrational rotation number \(\sigma\) to the rotation \(R_{\sigma}\), i.e., \(h^{-1}fh = R_{\sigma}\). Note that if \(h\) is known, then timeseries can be reconstructed because \(f^m(x) = h(h^{-1}x + m\sigma)\). It will be important to extend the notion of such a conjugacy to deal with the function pairs of our renormalisation group.

**Definition.** Let \((\xi, \eta)\in\mathcal{S}_n\) be such that \(\rho(\xi, \eta) = \sigma\). A **conjugating homeomorphism** of \((\xi, \eta)\) is a homeomorphism \(h: (\sigma - 1, \sigma) \to (\eta(0), \xi(0))\) such that \(h(0) = 0\) and

\[eh(x) = h(x + \sigma). \quad \sigma - 1 < x < 0\].
viewpoint were Hamiltonian systems, in particular area preserving maps, in action-angle variables $(r, \phi)$ [14, 28, 29]. The spectra of the homeomorphism that conjugates the phase variable along a particular critical K.A.M. surface to a pure rotation closely resembles the one computed here for cubic critical maps. There is a natural way to generate a critical circle map in the area preserving problem by simply restricting the 2-D map to the invariant K.A.M. curve. For the golden mean this map is found numerically to be $C^1$ and have no critical points.

Since the map is clearly critical by virtue of its singular invariant measure it is natural to ask whether under $T$ it would be attracted to any of the fixed points associated with inflectional circle maps, possibly non-analytic ones, e.g., (2.11). The answer is probably no, on empirical grounds.

We varied the inflectional exponent $b$ in (2.11) so as to obtain the same value of $\alpha$ as in ref. 29. We found $b = 1.82047$ gave $\alpha = 1.41485$. The spectra however did not agree. Whether anything can be said about the Hamiltonian problem from knowledge of the dissipative case remains to be seen.

It is worth noting that the fixed point we have found in dissipative systems is much more robust than can possibly be true for conservative systems. Whereas universality has been demonstrated for one degree of freedom area-preserving maps [28, 29], a continuum system will possess an infinity of frequencies. It is only through imposition of external fields or boundary conditions that a few degrees of freedom can be singled out. We know from laboratory experiment and rigorous mathematics that continuum dissipative systems can have low dimensional attractors (e.g., 2-tori).

The spectrum of quasi-periodic Hermitian operators (e.g., the Schrödinger equation with a quasi-periodic potential) is another small divisor problem of current interest [30]. Self-similarity in the distribution of band gaps as a function of winding number for a particular "critical" potential was noted by Hofstadter [31]. The problem of searching for extended states is equivalent to constructing a $(n + 1)$-torus in a nonlinear system for which in certain cases $n$ of the phases evolve according to a free rotation with mutually incommensurate frequencies. When $n = 2$ our renormalization group may be used to evaluate the string of matrix products (which each depend on a phase, $\phi$, on which $T$ acts), whose trace determines the Liapunov exponent of a particular state. One then generates a renormalisation transformation on the matrices (e.g., when $\phi_i$ lies in the interval removed by $T$ multiply $M(\phi_i)$ with its predecessor $\phi_{i-1}$ and call it $M^2(\phi_{i-1})$).

After $i$ applications one reduces the original matrix string to products of $M^n, M^{n-1}$. This grouping of the matrix multiplications is identical to the decimation scheme introduced in a different context by Feigenbaum and Hasslacher [32].

Finally we note that Manton and Nauenberg [33] have observed self-similar and singular behaviour at the limiting radius of convergence for the Schrödinger problem.

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**Appendix A**

*Transition from quasi-periodicity via phase-locking*

We discuss here some details concerning the ways in which phase-locked invariant tori can be destroyed and then relate this to the case studied above. Firstly, we consider a prototypical example involving the creation of a *critical cycle*. Then using the computer study of Aronson, Chory, Hall
and McGehee [34] we give a heuristic discussion of the bifurcation structure within a single phase-locked region. Much of this structure can be related to (not necessarily invertible) maps of the circle onto itself. After discussing this, we attempt to relate this picture to the case where the system is not phase-locked, as studied above.

Consider a two-parameter family \( P_{a,\omega} \) of diffeomorphisms of the annulus \( A \) into itself which contracts areas uniformly and where the nonlinear coupling \( a \) and the rotation \( \omega \) enter as in the family (1.1) discussed in the introduction, i.e., as the relevant parameters of our renormalisation group.

We first consider the bifurcations observed when \( a \) is small. To do this we need to consider the structure of the saddle-node bifurcation in a little more detail.

Consider a one-parameter family \( P_{\mu} \) of such mappings (e.g., \( P_{\mu} = P_{a,\omega(a,\mu)} \)). The conditions for a fixed point \( x \) of \( P_{\mu} \) to undergo a saddle-node bifurcation at \( \mu = 0 \) are the following: (a) the derivative \( dP_0(x_0) \) of \( P_0 \) at \( x_0 \) has 1 as a simple eigenvalue and no other eigenvalues of modulus equal to 1; (b) for \( |\mu| \) small there are coordinates \( (s, z, \mu) \in \mathbb{R} \times \mathbb{R} \times (-1, 1) \) on a neighbourhood of \( (x_0, 0) \) in \( A \times (-1, 1) \) such that \( (x_0, 0) \) is the origin and in this neighbourhood

\[
P_{\mu}(x, z) = (g_{\mu}(s), k_{\mu}(z)),
\]

where \( g_0(0) = 0, \ g'_0(0) = 1, \ g''_0(0) \neq 0, \ \frac{\partial g_0(0)}{\partial \mu}|_{\mu=0} \neq 0 \) and \( |dk_0(0)| < 1 \). Then the family \( g_{\mu} \) can be conjugated with the family \( \tilde{g}_{\mu} : \mathbb{R} \rightarrow \mathbb{R} \) given by \( \tilde{g}_{\mu}(s) = s + s^2 + \mu \). Thus the mapping \( (s, z, \mu) \rightarrow (s + s^2 + \mu, z/2, \mu) \) is a universal local model for the saddle-node.

At \( \mu = 0 \), locally near \( x_0 \) the situation is as shown in fig. 18. Near \( x_0 \), the outset \( \{ x : P_0^{-n}x \rightarrow x_0 \text{ as } n \rightarrow \infty \} \) consists locally of the curve \( \{ z = 0, s > 0 \} \); near \( x_0 \), the inset \( \{ x : P_0^n x \rightarrow x_0 \text{ as } n \rightarrow \infty \} \) consists of the half-space \( \{ s \leq 0 \} \). Moreover, there is a strong stable foliation of the local inset defined as follows: it is the partition of the inset into curves (called strong stable leaves) defined by the following rule: \( x \) and \( y \) lie in the same leaf if and only if \( \text{dist}(P_0^n x, P_0^n y) \) converges to zero exponentially fast as \( n \rightarrow \infty \). This foliation is invariant under \( P_0 \) in the sense that the \( P_0 \)-image of a leaf is contained in a leaf. Moreover, along \( z = 0 \), the tangents to these leaves depend continuously upon the base point. In some neighbourhood of \( x_0 \) the leaves are all transverse to the centre manifold \( z = 0 \). For a proof of all these facts see Newhouse, Palis and Takens [35]. The saddle-node bifurcation for periodic orbits is covered by the above description (if the period is \( q \), replace \( P \) by \( P^q \)).

We now consider the sequence of bifurcations involved in phase-locking when \( a \) is small and the invariant circle is preserved (see fig. 19). (Again, without loss of generality we can restrict to the case of a fixed point.) The bifurcations at \( \mu = \mu_1 \) and \( \mu_2 \) are saddle nodes. We shall only consider the bifurcation at \( \mu = \mu_1 \). Assume \( \mu_1 = 0 \). Here the invariant circle is the outset of the saddle-node. Since the outset is contained in the inset we have a cycle and this will force the bifurcation to have a non-trivial global structure even though the local structure (at the fixed point) is simple.

Assume that the outset \( \mathcal{O} \) is transverse to each strong stable leaf. Then using invariant manifold theory (Hirsch, Pugh and Shub [3]) one can prove that \( \mathcal{O} \) is smooth and normally hyperbolic. Therefore for small \( \mu \geq 0 \) there are smooth invariant circles \( \mathcal{O}_\mu \) near \( \mathcal{O} = \mathcal{O}_0 \) which depend smoothly upon \( \mu \).

At \( \mu = 0 \), \( P_0 \) has a fixed point (on \( \mathcal{O}_0 \)). For small \( \mu > 0 \), \( P_\mu \) has no fixed point. Therefore for all \( \epsilon > 0 \) there exists a set \( K \) in \((0, \epsilon)\) of positive Lebesgue measure such that if \( \mu \in K \) then the rotation number is irrational (Herman [8]). One expects that generically the closure of \( K \) is a Cantor set. If \( \mu \in K \) the
Fig. 19. The bifurcations creating and destroying a phase-locked (periodic) attractor on the invariant circle while staying within the quasi-periodic regime (i.e., preserving the invariant circle).

The attractor is the whole circle $C_p$. Thus we see that even in this controlled situation there is an explosion from a point attractor to a circle attractor. Nevertheless, for $\mu > 0$, the behaviour is still quasi-periodic. Thus we see that to destroy the quasi-periodicity we must not have transversality of $C_p$ to the strong stable foliation.

Consider again the sequence of bifurcations shown in fig. 19 above. We discuss the simplest change to this which produces a bifurcation away from quasi-periodicity. This is achieved if at some intermediate value of $\mu$, say $\mu = \mu_2$, $\mu_0 < \mu_2 < \mu_1$, the invariant circle loses its smoothness by becoming tangent to the (local) strong stable foliation defined near the sink so that for $\mu_2 < \mu < \mu_1$ the circle has quadratic tangencies with some of the leaves of this foliation. (The foliation is defined as follows. Assume that the eigenvalues of the sink are such that $0 < \lambda_1 < \lambda_2 < 1$. Then define the leaves by the condition that two points $x$ and $y$ lie in the same leaf if $\text{dist}(P^nx, P^ny) \approx \lambda_i^i$.) The leaf through the sink is called the strong stable manifold.

Then the important point to note is that at $\mu = \mu_1 = 0$, $C_p$ is tangent to the strong stable foliation of the saddle-node and the tangencies are quadratic (fig. 20). Then (a) for small $\mu > 0$ there is no invariant circle, (b) for all $\epsilon > 0$ there is a $\mu$ in $(0, \epsilon)$ such that $P_\mu$ has a horseshoe, and (c) there exists a sequence $\mu_i > 0$ such that $\lim_{i \to \infty} \mu_i = 0$ and $P_{\mu_i}$ has a homoclinic tangency for each $i$ [35].

Thus one sees that here the dynamical behaviour

Fig. 20. The simplest sequence of bifurcations creating and destroying the phase-locking and destroying the invariant circle. At $\mu = \mu_0$ there is a generic saddle-node creating the (phase-locked) periodic solution. At $\mu = \mu_2$, the outset of the saddle becomes (cubically) tangent to the strong stable foliation of the sink. For $\mu_2 < \mu < \mu_1$, the outset is quadratically tangent to this foliation. At $\mu = \mu_1$ the outset is quadratically tangent to the foliation defined by the saddle node point.
for $\mu > 0$ is very different from that in the previous bifurcation. In that case there was always an essentially global attractor which was periodic or quasi-periodic. In this case for many $\mu > 0$ there are infinitely many periodic orbits and also one can expect that associated to the homoclinic tangencies will be the Newhouse phenomenon of infinitely many sinks (Newhouse [36]) and behaviour similar to that of the Hénon attractor (Hénon [37]).

Now we consider all these bifurcations in a more global context. Recall that $I_{p/q}$ is the closure of the set of $(\omega, \alpha)$ such that $P_{\omega, \alpha}$ has an attracting periodic orbit (sink) whose rotation number is $p/q$. We now turn to consider the structure of $P_{\omega, \alpha}$ when $(\omega, \alpha) \in I_{p/q}$. The essential ideas are again captured by the simplest case where $p = 0$ and $q = 1$; i.e. a fixed point. Then our picture of what we believe happens in $I_{0/1}$ is summarised in fig. 21. The discovery of such a structure is due to Aronson, Chory, Hall and McGehee [34] who discuss this in detail for a particular two-parameter family and for the case where $p/q = 1/8$. We have only drawn one half of $I_{0/1}$ because one expects that what is happening in the other half is qualitatively similar and related by a symmetry. This is what is happening in the various regions and curves marked:

**Along BA:** The fixed point undergoes a saddle-node bifurcation.

**In I:** The invariant circle is smooth, and in the interior of I it contains a sink and a saddle. The strong stable manifold $W_{\text{sink}}$ of the sink and the inset or stable manifold $W_{\text{sad}}$ of the saddle intersect the boundary of the annulus in the way shown.

**On $\alpha$:** The outset of the saddle has become cubically tangent to the local strong stable foliation of the sink.

**In II:** The outset of unstable manifold of the saddle is now quadratically tangent to the sink’s local strong stable foliation, but does not intersect $W_{\text{sink}}$.

**On $\beta$:** The outset is quadratically tangent to $W_{\text{sink}}$.

**In III:** The outset intersects $W_{\text{sink}}$ transversally, but does not intersect $W_{\text{sad}}$.

**On $\gamma$:** The outset intersects $W_{\text{sink}}$ transversally and has a quadratic tangency with $W_{\text{sad}}$.

**In IV and VII:** The outset intersects $W_{\text{sink}}$ and $W_{\text{sad}}$ transversally.

**On $\delta$:** The outset is quadratically tangent to $W_{\text{sink}}$ as shown, and moreover, $W_{\text{sink}}$ does not intersect one of the boundary curves of the annulus.

**In V:** The outset of the saddle does not intersect $W_{\text{sink}}$. Both branches of $W_{\text{sink}}$ cross the same boundary curve of the annulus.

**On $\phi$:** As in V, except that the outset of the saddle is now quadratically tangent to its inset.

**In VI:** As on $\phi$ except that the outset now intersects the inset transversally.

**On $\tau$:** The outset is again quadratically tangent to the inset, but with the opposite orientation to that occuring in $\phi$.

**In VII:** Both $W_{\text{sink}}$ and $W_{\text{sad}}$ cross to one side of the annulus and the saddle’s outset does not intersect $W_{\text{sink}}$ and $W_{\text{sad}}$. Moreover, $W_{\text{sad}}$ separates the annulus into two invariant region.

**In IX:** The sink has complex conjugate eigenvalues. In crossing from I to IX the real eigenvalues of I become equal, then complex conjugate.

**In X:** These eigenvalues become real again.

**On crossing from X into XI:** There is a generic period-doubling bifurcation, and presumably a cascade of these as $\alpha$ is increased.

We note that the fixed point remains an attractor throughout the regions I to VIII but is not necessarily unique in III to VIII.

Now one can see why the global excitation caused by a saddle-node with a critical cycle will be ubiquitous in systems depending upon one parameter $\mu$.

Suppose that the system starts from some phase-locked situation as in I. If it leaves I by crossing BA then the system remains quasi-periodic. Another possibility is that it leaves I by crossing $\alpha$ in II. If the path followed by the system then leaves II by crossing BA transversally one observes precisely the bifurcation discussed above. Thus, in this scheme, this bifurcation is the simplest mechanism for a bifurcation from quasi-periodicity to chaos.

We now discuss the relationship with maps of the circle. Each of the regions in fig. 21 except V, VI and VII has a counterpart in, for example, the following two-parameter family of maps of the
Fig. 21. Some details of the summed bifurcation structure in the tongue $I_{q1}$. See the text for a description of the various regions.
circle:
\[ f_{\omega,a}(\theta) = \theta + \omega - \left(\frac{a}{2\pi}\right) \sin 2\pi \theta, \]

though one has to be careful in the exact interpretation of this correspondence. Essentially, the correspondence is obtained by thinking of the circle mappings as the infinite-dissipation limit of the annulus mappings. All points are mapped onto a circle after one iteration; the contraction in the radial direction is infinite. On taking this limit the outset of the saddle-point converges to the circle. The strong stable foliations converge to the foliation by radial lines and points at which the outset is tangent to a stable leaf converge to images of the critical points of the circle mapping. (Of course, the sinks of the circle and annulus mappings correspond, and the saddle-point of the annulus mapping corresponds to the unstable fixed point of the circle mapping.)

We have discussed the family \( f_{\omega,a} \) for \( 0 \leq a \leq 1 \) in the previous sections. Let \( I_{0/1}^{''} \) denote the closure of the region of \((\omega, a)\)-space in which \( f_{\omega,a} \) has a sink (i.e., an attracting fixed point). The region consisting of the intersection of \( I_{0/1}^{''} \) with \( 0 < a < 1 \) corresponds to \( I \). The curve in \( I_{0/1}^{''} \) where \( a = 1 \) corresponds to the upper boundary of \( I \), so that the mappings in II and IX are respectively analogous to circle mapping like (a) and (b) in fig. 22. In fig. 22a the critical points \( c_1 \) and \( c_2 \) are such that \( f_{\omega,a}^{i}(c_i), \ i = 1, 2, \) converge monotonically (i.e., from side to side) to the sink as \( n \to \infty \). In fig. 22b they converge to the sink, but in an oscillatory manner. These properties correspond respectively to the fact that in I and II the outset of the saddle converges monotonically to the sink while in IX it spirals towards it. In III the outset has penetrated the strong stable manifold of the sink. The analogous property for the circle mappings is that the sink lies between \( f_{\omega,a}(c_1) \) and \( f_{\omega,a}(c_2) \) in in fig. 22c. (The stacked folds that bracket \( W_{\text{sink}} \) in region III of fig. 21 correspond to images of the two critical points.) Any annulus mapping sufficiently close to such a circle mapping will be like those in III. The circle mappings corresponding to \( \gamma \) have the property that \( f_{\omega,a}(c_1) \) is the unstable fixed point. This corresponds to the fact that on \( \gamma \) the outset is tangent to the inset of the saddle. Clearly, the annulus mappings in, for example, V, have no similar 1-dimensional analogue because the strong stable manifold of the sink only meets one boundary curve of the annulus, whereas in the infinite
dissipation situation this curve is a radial arc and hence meets both boundary curves of the annulus. In particular, we see that the curve \( \gamma' \) consisting of those \((\omega, a)\) such that \( f_{\omega,a}(e_1) \) is the unstable fixed point (i.e., the analogue of \( \gamma \)) extends across \( I_{b}\) as far as the region analogous to \( I_{X} \). Below \( \gamma' \) the rotation number,

\[
\rho(f, \theta) = \limsup_{n \to \infty} n^{-1}(f^n \theta - \theta)
\]

of \( f = f_{\omega,a} \) at \( \theta \) is independent of \( \theta \) and exists as a limit; while above \( \gamma \) it depends upon \( \theta \).

In fact, identifying numbers which differ by an integer, it can be shown that

\[
R(f) = \{\rho(f, \theta) : 0 \leq \theta \leq 1\}
\]

is a closed interval \([\rho_-(f), \rho_+(f)]\) and that \( \rho_- \) and \( \rho_+ \) are continuous functions of \( f \) (Newhouse, Palis and Takens [35], Ito [39]). Above \( \gamma' \), \( \rho_-(f) \neq \rho_+(f) \). In particular, there are some points \( \theta \) with irrational rotation number.

Consider now the general structure of such mappings. In contrast to the diffeomorphism case (i.e., when \( 0 < a < 1 \) for the \( f_{\omega,a} \)), when \( a > 1 \), the mapping \( f = f_{\omega,a} \) always has a periodic orbit (Block and Franke [38]); thus \( R(f) \) always contains a rational number. In fact, if \( \rho_-(f) \neq \rho_+(f) \) then there are infinitely many unstable periodic orbits for \( f \), because if \( p/q \in R(f) \) then it is easy to show that one can solve the equation \( f^n \theta = \theta + p \) and, moreover, the solution \( \theta_0 \) can be chosen so that if \( U \) is any neighbourhood of \( \theta_0 \) in the circle then \( U, fU, \ldots, f^n U \) cover the circle for some \( n > 0 \) (Newhouse, Palis and Takens [35]). On the other hand \( f = f_{\omega,a} \) has at most two stable periodic orbits, and if there are two then these contain the critical points in their basins of attraction. Moreover, from work of Jacobson [40], one expects that such mappings are structurally stable (and hence the \( \rho_+ \) and \( \rho_- \) are locally constant) if and only if the critical points are contained within the basins of stable periodic orbits. Newhouse, Palis and Takens [35] show that if \( f_{\mu} \) is a generic one-parameter family and if \( \mu_0 \) is such that not both of \( \rho_+(f_{\mu}) \) and \( \rho_-(f_{\mu}) \) are locally constant at \( \mu_0 \) then in every neighbourhood of \( \mu_0 \) there is a point \( \mu \) such that for some \( n > 0 \) the \( f^n \)-image of a critical point is an unstable periodic point. This is the one-dimensional (infinite dissipation) analogue of a homoclinic tangency.

Now consider the set \( F(\sigma_G) \) consisting of those points \((\omega, a)\) for which \( \rho(f_{\omega,a}, \theta) = \sigma_G = (\sqrt{5} - 1)/2 \) for some \( \theta \). As we have seen, for \( 0 < a < 1 \) this is a curve of the form \( \omega = u(a) \), \( u \) analytic. For \( a > 1 \), if \((\omega, a) \in F(\sigma_G)\) then \( \rho_-(f_{\omega,a}) \neq \rho_+(f_{\omega,a}) \) because \( R(f_{\omega,a}) \) contains a rational and an irrational. We can approximate \( F(\sigma_G) \) by the regions \( I_{p/q} \). Therefore we expect that \( F(\sigma_G) \) will have the form depicted in fig. 23. We conjecture (a) that the boundary curves of \( F(\sigma_G) \) are the limit as \( n \to \infty \) of boundaries of \( I_{p/q} \) and \( I_{p/q + 1} \), and (b) that for \((\omega, a)\) in this boundary curve there is a \( f_{\omega,a} \)-invariant Cantor set \( A \) which (i) is the closure of the orbit of a critical point, (ii) is attracting, and (iii) has the property that if \( \theta \in A \) then \( \rho(f_{\omega,a}, \theta) = \sigma_G \). (Presumably, an invariant Cantor set satisfying (iii) exists if \((\omega, a)\) lies in the interior of \( F(\sigma_G) \), but then it would not be an attractor.)
References

[11] The set A is defined by restricting the growth rate of the entries in the continued fraction (2.3). Specifically \( \sigma \in A \Leftrightarrow \lim_{N \to \infty} \limsup_{M \to \infty} \sum_{\eta_n > M} \ln(1 + \eta_n) / \sum_{1 \leq n \leq N} \ln(1 + \eta_n) = 0. \)

The set A has Lebesgue measure one on the interval, and for all \( \epsilon > 0 \) there exists a C, such that

\[ |\sigma - p/q| \geq C_q^{-\epsilon}. \]